Semilinear non-autonomous problems with unbounded coefficients in the linear part

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This paper is devoted to the basic theory of a class of semilinear nonautonomous parabolic problems with non standard linear part. Namely, we consider Cauchy problems such as

$$\begin{cases} D_t u(t,x) = (\mathcal{A}(t)u)(t,x) + \psi(t,u(t,x)), \\ \qquad\qquad t > s,\ x \in \mathbb{R}^d, \\ u(s,x) = f(x), \quad x \in \mathbb{R}^d, \end{cases} \tag{1}$$

where the elliptic operators

$$\mathcal{A}(t) := \sum_{i,j=1}^d q_{ij}(t,x)D_{ij} + \sum_{i=1}^d b_i(t,x)D_i$$

have unbounded coefficients q_{ij}, b_i in $I \times \mathbb{R}^d$, I being a right halfline or the whole $\mathbb{R}$, $D_i = \partial/\partial x_i$, $D_{ij} = \partial^2/\partial x_i \partial x_j$.

We make suitable assumptions on the coefficients in such a way that the linear part generates a Markov evolution operator $G(t,s)$ in $C_b(\mathbb{R}^d)$, the space of the bounded and continuous functions from $\mathbb{R}^d$ to $\mathbb{R}$. The coefficients of $\mathcal{A}(t)$ are smooth enough, namely locally $C^{\alpha/2,\alpha}$ for some $\alpha \in (0,1)$, the matrices $Q(t,x) = [q_{ij}(t,x)]_{i,j=1,\dots d}$ are uniformly positive definite, and there exists a C^2 Lyapunov function $\varphi : \mathbb{R}^d \mapsto [0,+\infty)$ such that $\lim_{|x|\to+\infty} \varphi(x) = +\infty$ and

$$(\mathcal{A}(t)\varphi)(x) \le a - c\,\varphi(x), \quad (t,x) \in I \times \mathbb{R}^d,$$

for some positive constants a and c. Such assumption allows to use maximum principle arguments both in linear and in nonlinear equations. The evolution operator $G(t,s)$ is a contraction in $C_b(\mathbb{R}^d)$, namely

$$\|G(t,s)f\|_\infty \le \|f\|_\infty, \quad f \in C_b(\mathbb{R}^d),$$

and for any $s \in I$, $(t,x) \mapsto (G(t,s)f)(x) \in C^{1,2}((s,+\infty)\times\mathbb{R}^d) \cap C([s,+\infty)\times\mathbb{R}^d)$ is the unique bounded solution of

$$\begin{cases} D_t v(t,x) = (\mathcal{A}(t)v)(t,x), \quad t > s,\ x \in \mathbb{R}^d, \\ v(s,x) = f(x), \quad x \in \mathbb{R}^d. \end{cases}$$

Further assumptions allow to get global smoothing properties of $G(t,s)$ similar to the case of bounded coefficients,

$$\|\nabla_x G(t,s)f\|_\infty \le \frac{C_1}{\sqrt{t-s}}\|f\|_\infty, \quad f \in C_b(\mathbb{R}^d),$$

$$\|\nabla_x G(t,s)f\|_\infty \le C_2\|f\|_{C^1_b(\mathbb{R}^d)}, \quad f \in C^1_b(\mathbb{R}^d),$$

uniformly for $s < t$ in bounded intervals.

The construction of the evolution operator $G(t,s)$ and its main properties are in [2]. They are used to get local and global existence and uniqueness results for (1) when $f \in C_b(\mathbb{R}^d)$, under standard assumptions on ψ.

The case of L^p initial data is more difficult. Even in the linear autonomous case $\mathcal{A}(t) \equiv \mathcal{A}$, the Cauchy problem may be not well posed in $L^p(\mathbb{R}^d, dx)$ if the coefficients of $\mathcal{A}$ are unbounded, unless the coefficients satisfy very restrictive growth assumptions. The only way to work in L^p spaces is to replace the Lebesgue measure dx by another measure, possibly a weighted measure $\rho(x)dx$. The best situation in the autonomous case is when there exists an invariant measure μ, namely a Borel probability measure such that

$$\int_{\mathbb{R}^d} T(t)f\,d\mu = \int_{\mathbb{R}^d} f\,d\mu, \quad t > 0,\ f \in C_b(\mathbb{R}^d),$$

where $T(t)$ is the Markov semigroup associated to $\mathcal{A}$ in $C_b(\mathbb{R}^d)$. Under reasonable assumptions, a unique invariant measure exists, it is absolutely continuous with respect to the Lebesgue measure, and it is related to the asymptotic behavior of $T(t)$, since

$$\lim_{t\to+\infty}(T(t)f)(x) = \int_{\mathbb{R}^d} f\,d\mu,$$

for any $f \in C_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Moreover, the operators $T(t)$ are easily extended to contractions in the spaces $L^p(\mathbb{R}^d, \mu)$ for every $p \in [1,+\infty)$.

The nonautonomous case is more complex. In general, a measure μ such that

$$\int_{\mathbb{R}^d} G(t,s)f\,d\mu = \int_{\mathbb{R}^d} f\,d\mu, \quad t > s,\ f \in C_b(\mathbb{R}^d),$$

does not exist. What plays the role of invariant measures are the evolution systems of measures, namely families of Borel probability measures $\{\mu_t : t \in I\}$ such that

$$\int_{\mathbb{R}^d} G(t,s)f\, d\mu_t = \int_{\mathbb{R}^d} f\, d\mu_s,$$

for any $t > s \in I$ and $f \in C_b(\mathbb{R}^d)$. In this case, $G(t,s)$ is readily extended to a contraction from $L^p(\mathbb{R}^d, \mu_s)$ to $L^p(\mathbb{R}^d, \mu_t)$ for $t > s$, for every $p \in [1, +\infty)$. However, in contrast to the autonomous case, where the invariant measure is unique under very weak assumptions, evolution systems of measures are not unique. Among all evolution systems of measures, the one related to the asymptotic behavior of $G(t,s)$ is the (unique) tight[(1)] evolution system of measures. See [2,1].

In the paper [2] a tight evolution system of measures $\{\mu_t : t \in I\}$ was proved to exist. Here we set our nonlinear problem in the spaces $L^p(\mathbb{R}^d, \mu_t)$ where $\{\mu_t : t \in I\}$ is such a tight evolution system of measures. As usual, to work in a L^p context, the nonlinearity is assumed to be Lipschitz continuous with respect to u. We introduce the measure ν in $I \times \mathbb{R}^d$, defined by

$$\nu(J \times \mathcal{O}) := \int_J \mu_t(\mathcal{O})\, dt,$$

on Borel sets $J \subset I$, $\mathcal{O} \subset \mathbb{R}^d$ and canonically extended to the Borel sets of $I \times \mathbb{R}^d$. For every $f \in L^p(\mathbb{R}^d, \mu_s)$ we prove existence in the large and uniqueness of a solution u to (1) belonging to $L^p((s,\tau) \times \mathbb{R}^d, \nu)$, for every $\tau > s$. Moreover, $\sup_{s<t<\tau} \|u(t,\cdot)\|_{L^p(\mu_t)} < \infty$.

By "solution" to (1) in an interval $[s,\tau]$ we mean a mild solution, namely a function that satisfies the identity

$$u(t,\cdot) = G(t,s)f + \int_s^t G(t,r)\psi(r, u(r,\cdot))dr,$$

for any $s \le t \le \tau$. If f is continuous and bounded, the mild solution is shown to be a classical solution and sufficient conditions for existence in the large are provided. If $f \in L^p(\mathbb{R}^d, \mu_s)$, the mild solution is a strong solution in the sense of Friedrichs, namely it is the L^p-limit of a sequence of classical solutions.

We study also asymptotic behavior results. Assuming that $\psi(t,0) = 0$ for every t, we prove a nonautonomous version of the principle of linearized stability in the space $C_b(\mathbb{R}^d)$. In addition, under a very strong dissipativity assumption on ψ,

$$\xi\, \psi(t,\xi) \le \psi_0\, \xi^2, \qquad t \in I,\ \xi \in \mathbb{R},$$

with $\psi_0 \le 0$, we prove a global stability result: for every $f \in C_b(\mathbb{R}^d)$, the solution u to (1) satisfies

$$|u(t,x)| \le e^{\psi_0(t-s)}\|f\|_\infty, \quad t > s,\ x \in \mathbb{R}^d.$$

The same assumption, together with some technical assumptions on the growth of the coefficients as $|x| \to \infty$, allows to prove a similar result in our L^p context: for every $f \in L^p(\mathbb{R}^d, \mu_s)$, the solution u to (1) satisfies

$$\|u(t,\cdot)\|_{L^p(\mu_t)} \le e^{\psi_0(t-s)}\|f\|_{L^p(\mu_s)}, \quad t > s. \tag{2}$$

If the measures μ_t satisfy a uniform logarithmic Sobolev type inequality with constant K,

$$\int_{\mathbb{R}^d} |g|^\gamma \log|g|\, d\mu_r \le \|g\|^\gamma_{L^\gamma(\mu_r)} \log\|g\|_{L^\gamma(\mu_r)} + \gamma K \int_{\{g\neq 0\}} |g|^{\gamma-2}|\nabla g|^2 d\mu_r, \tag{3}$$

for any $r \in I$, $g \in C^1_b(\mathbb{R}^d)$ and $\gamma \in (1, +\infty)$, then estimate (2) can be improved as follows,

$$\|u(t,\cdot)\|_{L^{p(t)}(\mu_t)} \le e^{\psi_0(t-s)}\|f\|_{L^p(\mu_s)}, \quad t > s,$$

where $p(t) := e^{\eta_0 K^{-1}(t-s)}(p-1)+1$, η_0 being the ellipticity constant. So, we get a hypercontractivity property that is similar to the linear case ([1]) if $\psi_0 = 0$, hypercontractivity plus exponential decay if $\psi_0 < 0$. Sufficient conditions for the occurrence of the logarithmic Sobolev inequalities (3) are in [1].

The results in C_b are obtained adapting the usual techniques of semilinear parabolic equations (e.g., [3,4]) to our situation.

The results in L^p are much less straightforward. In particular, (1) cannot be seen as an evolution equation in a fixed L^p space, because our spaces $L^p(\mathbb{R}^d, \mu_t)$ may depend explicitly on t.

Several examples of operators $\mathcal{A}(t)$ that satisfy our assumptions are in the papers [2,1] to which we refer for detailed proofs.

[1] A set of Borel measures $\{\mu_t : t \in I\}$ in $\mathbb{R}^d$ is tight if for every $\varepsilon > 0$ there exists $\rho > 0$ such that $\mu_t(\mathbb{R}^d \setminus B(0,\rho)) \le \varepsilon$, for every $t \in I$.

REFERENCES

1. L. Angiuli, L. Lorenzi, A. Lunardi, Comm. Part. Diff. Eqns.,28, (2013), 2049.
2. M. Kunze, L. Lorenzi, A. Lunardi, Trans. Amer. Math. Soc. 362 (2010) 169.
3. O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'ceva, Linear and quasilinear equations of parabolic type, Nauka, Moscow, 1967.
4. D. Henry, Geometric theory of semilinear parabolic equations, Lect. Notes in Math., Springer-Verlag, New York (1981).