Non autonomous parabolic problems with unbounded coefficients in unbounded domains

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Parabolic Cauchy problems with unbounded coefficients set in unbounded domains, with sufficiently smooth boundary, have been studied in the autonomous case both in the case of homogeneous Dirichlet and Neumann boundary conditions. On the other hand, the nonautonomous counterpart have been studied, to the best of our knowledge, only in the particular case $\Omega = \mathbb{R}^d_+$, again only under homogeneous Dirichlet and Neumann boundary conditions [2].

This paper is devoted to continue the analysis started in [2], studying parabolic nonautonomous boundary Cauchy problems with unbounded coefficients in a greater generality, with respect to both the domain, where the Cauchy problems are set, and the boundary conditions considered. More precisely, let $\Omega \subset \mathbb{R}^d$ be an unbounded open set with a boundary of class $C^{2+\alpha}$, for some $\alpha \in (0,1)$, and let $I \subset \mathbb{R}$ be an open right halfline (possibly $I = \mathbb{R}$). For any fixed $s \in I$ and any $f \in C_b(\Omega)$ (the space of bounded and continuous functions on Ω), we consider the nonautonomous Cauchy problem

$$\begin{cases} D_t u(t,x) = (\mathcal{A}u)(t,x), & t \in (s,+\infty),\ x \in \Omega, \\ (\mathcal{B}u)(t,x) = 0, & t \in (s,+\infty),\ x \in \partial\Omega, \\ u(s,x) = f(x), & x \in \Omega. \end{cases} \tag{$P_{\mathcal{B}}$}$$

where the operators $\{\mathcal{A}(t)\}_{t\in I}$ and $\{\mathcal{B}(t)\}_{t\in I}$ are defined as follows:

$$\mathcal{A}(t) = \sum_{i,j=1}^d q_{ij}(t,\cdot)D_{ij} + \sum_{i=1}^d b_i(t,\cdot)D_i - c(t,\cdot), \tag{1}$$

$$\mathcal{B}(t) = \sum_{i=1}^d \beta_i(t,\cdot)D_i + \gamma(t,\cdot), \quad t \in I. \tag{2}$$

The coefficients of the previous operators are smooth enough functions, and all of them but β may be unbounded; function β either everywhere differs from 0 on $\partial\Omega$ or therein identically vanishes. In the first case, we assume the usual non-tangential condition, in the latter one, we assume that $\gamma \equiv 1$ so that $\mathcal{B}\zeta$ is the trace of ζ on $\partial\Omega$.

We first prove existence and uniqueness of a bounded classical solution of problem ($P_{\mathcal{B}}$). The case $\gamma \geq 0$ requires rather weak assumptions on the coefficients of the operators $\mathcal{A}(t)$ and $\mathcal{B}(t)$. No growth assumptions are assumed on the diffusion and drift coefficients of the operators $\mathcal{A}(t)$, whereas the potential is assumed to be bounded from below to guarantee existence of bounded solutions to problem ($P_{\mathcal{B}}$). Further, the existence of a so-called Lyapunov function φ, associated with the pair $(\mathcal{A}(t), \mathcal{B}(t))$ is assumed, which serves as a fundamental tool to prove a maximum principle, which yields uniqueness of the solution to problem ($P_{\mathcal{B}}$). When γ takes also negative values we assume an extra condition, which is stated in terms of another Lyapunov function. The existence and the uniqueness of a classical solution to problem ($P_{\mathcal{B}}$) allow us to define an evolution operator $G_{\mathcal{B}}(t,s)$ of bounded linear operators in $C_b(\Omega)$ and to prove some remarkable continuity properties that this evolution operator enjoys. As a consequence of the Riesz representation theorem and the continuity property of the evolution operator, we can show that, for any $(t,s) \in \Lambda := \{(t,s) \in I \times I : t > s\}$ and any $x \in \Omega$, there exists a finite Borel measure $g_{\mathcal{B}}(t,s,x,dy)$ such that

$$(G_{\mathcal{B}}(t,s)f)(x) = \int_\Omega f(y) g_{\mathcal{B}}(t,s,x,dy), \tag{3}$$

for any $f \in C_b(\Omega)$. Under an additional smoothness assumption on the diffusion coefficients we prove that $G_{\mathcal{B}}(t,s)f$ admits an integral representation by means of a Green function $g_{\mathcal{B}} : \Lambda \times \Omega \times \Omega \to (0,+\infty)$, i.e., $g_{\mathcal{B}}(t,s,x,dy) = g_{\mathcal{B}}(t,s,x,y)dy$ for any $(t,s,x,y) \in \Lambda \times \Omega \times \Omega$. For any fixed $s \in I$ and almost any $y \in \Omega$, the function $g_{\mathcal{B}}(\cdot,s,\cdot,y)$ is smooth, satisfies $D_t g_{\mathcal{B}} - \mathcal{A}(t) g_{\mathcal{B}} = 0$ in $(s,+\infty) \times \Omega$.

Formula (3) plays a crucial role in the study of the compactness of the operators $G_{\mathcal{B}}(t,s)$ in

$C_b(\Omega)$, for $(t,s) \in \Lambda \times J^2$, J being a bounded interval which follows from the tightness of the family of measures $\{g_{\mathcal{B}}(t,s,x,dy),\, x \in \Omega\}$ for any $(t,s) \in \Lambda \cap J^2$. In view of this fact, a sufficient condition is then provided to guarantee the tightness of the previous family of measures. Our result extends the results obtained in [1,7] in the case when $\Omega = \mathbb{R}^d$.

Next, when the boundary operator $\mathcal{B}$ is independent of t, under some growth assumptions on the coefficients q_{ij}, b_i and c at infinity and assuming that they are bounded in a small neighborhood of $\partial\Omega$, we prove an uniform gradient estimate for $G_{\mathcal{B}}(t,s)f$. More precisely, we show that for any $T > s \in I$, there exists a positive constant $C_{s,T}$ such that

$$\|\nabla_x G_{\mathcal{B}}(t,s)f\|_\infty \leq \frac{C_{s,T}}{\sqrt{t-s}}\|f\|_\infty, \tag{4}$$

for any $f \in C_b(\Omega)$ and $t \in (s,T)$. Estimate (4) (which can be then extended, by the evolution law, to all $t \in (s,+\infty)$) is classical when the coefficients of $\mathcal{A}(t)$ are bounded and Ω is an open set with sufficiently smooth boundary, either bounded or unbounded (see [8]). Recently, it has been proved for the semigroup $T(t)$ associated in $C_b(\Omega)$ to autonomous elliptic operators with unbounded coefficients, both in the case of homogeneous Neumann (first in convex sets [4] and, then, in the general case [3]) and Dirichlet boundary conditions [5]. Very recently, we proved estimate (4) for the solution to problem $(P_{\mathcal{B}})$ in $\mathbb{R}^d_+$ when homogeneous Dirichlet and Neumann boundary conditions are prescribed on $\partial\mathbb{R}^d_+$. The simple geometry of $\mathbb{R}^d_+$ and suitable assumptions on the coefficients of the operator $\mathcal{A}(t)$, allowed to extend these latter ones to $\mathbb{R}^d$ and to reduce the problem to the whole space $\mathbb{R}^d$, where gradient estimates were already known ([6]). A symmetry argument was then used to come back to the Neumann and Dirichlet Cauchy problems set in $\mathbb{R}^d_+$.

In our situation the key tools to prove (4) are the Bernstein method and a geometric transformation which allows to locally transform the boundary Cauchy problem $(P_{\mathcal{B}})$ into a Cauchy problem in the halfspace $\mathbb{R}^d_+$ where homogeneous Robin boundary conditions are prescribed. Here, the idea is to use the regularity of the domain to go back by means of local charts to problems defined in $\mathbb{R}^d_+$ or in $\mathbb{R}^d$. Assuming more smoothness on the domain Ω and the vector β, we determine coordinate transformations which, locally transform the homogeneous boundary condition $\mathcal{B}u = 0$ on the boundary $\partial\Omega$ to an homogeneous Robin boundary condition on $\mathbb{R}^{d-1} \times \{0\}$. Thus, under the assumption that the coefficients of $\mathcal{A}(t)$ are bounded only in a neighborhood of the boundary $\partial\Omega$, we prove an uniform gradient estimates in a small strip Ω_δ near the boundary. Finally, some growth assumptions on the diffusion coefficients and the potential term and a quite standard dissipativity condition on the drift term b, are enough to show that (4) is satisfied also in $\Omega\backslash\Omega_\delta$. We point out that, differently from [4,3,5], we do not assume that the diffusion coefficients q_{ij} are globally bounded together with their spatial gradients. Moreover, our results seem to be new also in the autonomous case when $\mathcal{B}$ is a general first-order boundary operator. In particular, we can cover also the case when γ changes sign on $\partial\Omega$.

The special case when Ω is convex and homogeneous Neumann boundary conditions are prescribed, can be treated and estimate (4) can be proved without assuming any additional smoothness assumption on the domain and any hypotheses of boundedness for the coefficients of $\mathcal{A}(t)$ in a neighborhood of the boundary. This can be done adapting the arguments used in the autonomous case, described here above.

Also when $\Omega = \mathbb{R}^d_+$ and homogeneous Robin boundary conditions are prescribed on $\mathbb{R}^{d-1} \times \{0\}$, we do not need to assume that the drift term b and the potential term c are bounded. Indeed, a simple trick allows us to transform homogeneous Robin boundary condition into homogeneous Neumann condition on $\partial\mathbb{R}^d_+$. Hence, we are reduced to a problem set in a convex set with Robin boundary conditions, to which we can apply the already established results.

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