Nonlinear Schrödinger systems with nonzero boundary conditions

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The nonlinear Schrödinger (NLS) equation is a universal model for the behavior of weakly nonlinear, quasi-monochromatic wave packets, and they arise in a variety of physical settings. In particular, the so-called defocusing NLS equation:

$$iq_t + q_{xx} - 2|q|^2 q = 0 \tag{1}$$

describes the stable propagation of an electromagnetic beam in (cubic) nonlinear media with normal dispersion, and has been the subject of renewed applicative interest in the framework of recent experimental observations in Bose-Einstein condensates and dispersive shock waves in optical fibers.

The defocusing NLS, admits soliton solutions with nonzero boundary conditions (NZBCs), so-called dark/gray solitons, which have the form:

$$\begin{gathered} q(x,t) = q_0 e^{-2iq_0^2 t}\left[\cos\alpha + i\sin\alpha\,\tanh\xi\right] \\ \xi = q_0\sin\alpha\,(x + 2q_0\,t\cos\alpha - x_0) \end{gathered} \tag{2}$$

with q_0, α and x_0 arbitrary real parameters. Dark soliton solutions are such that $|q(x,t)| \to q_0$ as $x \to \pm\infty$, and appear as localized dips of intensity $q_0^2 \sin^2\alpha$ on the background field q_0.

Even though the inverse scattering transform (IST) as a method to solve the initial-value problem for the NLS equation was first proposed almost 40 years ago [1], with boundary conditions (BCs) taken as

$$q(x,t) \to q_\pm(t) = q_0 e^{-2iq_0^2 t + i\theta_\pm} \quad \text{as} \quad x \to \pm\infty\,,$$

and has been subsequently studied by several authors [2–6], many important issues still remain to be clarified. The main reason why, in spite of the deep experimental relevance of the problem, a complete and rigorous IST theory is still unavailable is due to the fact that one has to deal with solutions that do not decay at space infinity. This makes the IST significantly more involved than in the case of decaying potentials, in particular as far as the analyticity of the eigenfunctions of the associated scattering problem. As a matter of fact, (the analog of) Schwartz class is usually assumed for the potential, which is clearly unnecessarily restrictive. In [2] the issue of establishing the analyticity of the eigenfunctions was addressed by reformulating the scattering problem in terms of a so-called energy dependent potential, but the drawback of that approach is a very complicated dependence of eigenfunctions and data on the scattering parameter.

A recent step forward in the direction of a rigorous IST for the defocusing NLS with NZBCs was proposed in [9], where we proved that the direct scattering problem is well defined for potentials q such that $q - q_\pm \in L^{1,2}(\mathbb{R}^\pm)$, $L^{1,s}(\mathbb{R})$ being the complex Banach space of all measurable functions $f(x)$ for which $(1+|x|)^s f(x)$ is integrable, and analyticity of eigenfunctions and scattering data was proved.

As to the inverse problem, we formulated and solved it both via Marchenko integral equations, and as a Riemann-Hilbert problem in terms of a suitable uniform variable. We determined the asymptotic behavior of the scattering data and showed that the linear system solving the inverse problem is well defined. Finally, we developed the triplet method as a tool to obtain explicit multisoliton solutions by solving the Marchenko integral equation via separation of variables.

An important open issue is whether an area theorem can be established, to relate the existence and location of discrete eigenvalues of the scattering problem to the area of the initial profile of the solution, suitably defined to take into account the NZBCs.

For the focusing NLS with vanishing BCs, it was shown [8] that there are no discrete eigenvalues and no spectral singularities if the L^1-norm of the potential is smaller that $\pi/2$. Conversely, it is well-known that no such result holds for the Korteweg-de Vries equation, whose scattering problem [which is the time-independent Schrödinger equation] with positive initial datum can have discrete eigenvalues even for initial profiles with arbitrarily small area.

In [10] we proved that no area theorem exists for the defocusing NLS with nonzero boundary conditions, by providing explicit examples of box-type initial conditions where at least one discrete eigenvalue exists, no matter how small the difference between the initial profile and the background field is.

Specifically, we considered for the scattering problem associated to the defocusing NLS equation, which is nothing but the Dirac equation with non-zero rest mass, a piecewise-constant initial condition (IC) of the type

$$q(x,0) = \begin{cases} q_- = q_o\, e^{-i\theta} & x < -L\,, \\ q_c = h\, e^{i\alpha} & -L < x < L\,, \\ q_+ = q_o\, e^{i\theta} & x > L\,, \end{cases} \tag{3}$$

where h, q_o and L are arbitrary nonnegative parameters, and α and θ are arbitrary phases. The above IC models a potential well or a potential barrier (both on a non-zero background) when $h < q_o$ and when $h > q_o$, respectively.

When $\theta = 0$, the potential is an even function of x, and thus discrete eigenvalues come in opposite pairs. Thus, in this case one can restrict the search for eigenvalues to the range $0 < k < q_o$. It is convenient to introduce $r = h/q_o$ and $\omega q_o L$, and to rescale the scattering parameter as $k = q_o y$. Figs. 1 and 2 show the discrete spectrum in some examples discussed in the paper, to show that at least one discrete eigenvalue exists, no matter how small the difference between the initial profile and the background field is.

The second issue addressed in [10] is whether the radiative part of the spectrum can yield a nontrivial contribution to the asymptotic phase difference of the potential. It is well-known (see, for instance, [6]) that the trace formula for the NLS equation determines the asymptotic phase difference, $\arg(q_+/q_-)$, in terms of a contribution from the discrete spectrum and one from the continuous spectrum via the reflection coefficient. We showed that the radiative components of the solution can indeed provide a non-zero contribution to the asymptotic phase difference of the potential. Again, we do so by explicitly providing examples of piecewise constant initial conditions corresponding to a non-zero asymptotic phase difference in the potential for which no discrete eigenvalues are present.

From an applied point of view, these results are relevant in the context of recent theoretical studies and experimental observations of defocusing NLS in the framework of dispersive shock waves in optical fibers (see, for instance, [7] regarding the appearance and evolution of dispersive shock waves when an input (reflectionless) pulse containing a large number of dark or gray solitons

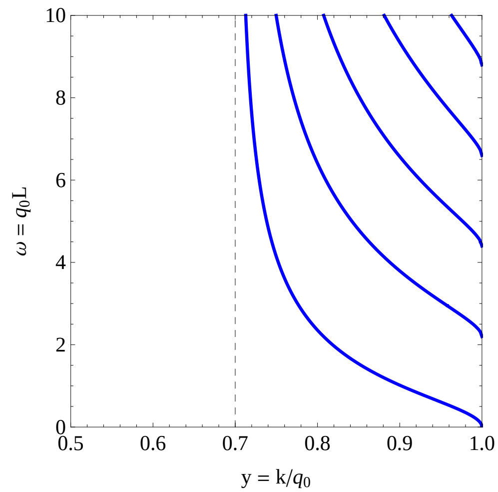

Figure 1. The discrete spectrum of the potential well as a function of ω for $r = 0.7$ and $\alpha = 0$. The horizontal axis is $y = k/q_o$, the vertical axis is ω. The dashed vertical line identifies the value $y = r$. The dotted horizontal lines delimit the exclusion zone $\sqrt{1-r^2}/[2r(r-\cos\alpha)] \le \omega \le \pi/(2\sqrt{1-r^2})$.

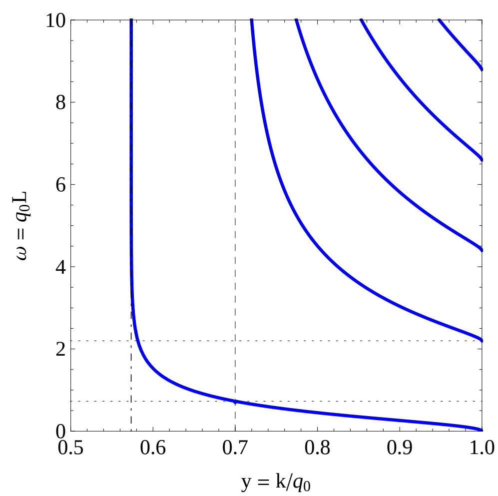

Figure 2. The discrete spectrum of the potential well as a function of ω for $r = 0.7$ and $\alpha = \pi/2$, axes as in Fig. 1.

is injected in the fiber). Moreover, this work will pave the way for generalizing similar results to the defocusing vector NLS equation.

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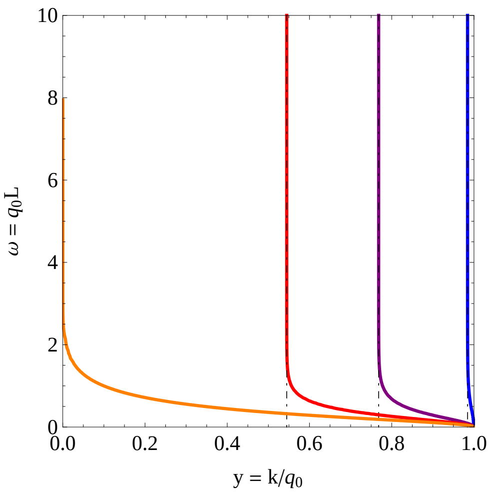

Figure 3. The discrete spectrum of the potential barrier ($r > 1$) as a function of ω for $r = 1.2$ and a few different values of α: $\alpha = \pi/4$ (blue), $\alpha = \pi/2$ (purple), $\alpha = 2\pi/3$ (red) and $\alpha = \pi$ (orange). As before, the horizontal axis is $y = k/q_o$, the vertical axis is ω.

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