Ordinary differential equations uniquely determined by their symmetries

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In the context of the geometric theory of symmetries of (systems of) differential equations (DEs) a natural problem is to see when a DE, either partial (PDE) or ordinary (ODE), is uniquely determined by its Lie algebra of point symmetries.

The research activity that we have performed is to investigate the inverse problem in the context of ODEs: given a Lie algebra $\mathfrak{s}$ of vector fields, how to construct ODEs having $\mathfrak{s}$ as a Lie point symmetry subalgebra and satisfying some specific properties that ensure the uniqueness of such DE.

The idea of describing DEs admitting a given Lie algebra of symmetries dates back at least to S. Lie, who stated that the $u_{xx} = 0$ is the unique scalar second order ODE, up to point transformations, admitting an 8-dimensional Lie algebra of symmetries. Several authors devoted their research to finding differential equations by specifying their symmetries or their conservation laws (see, *e.g.*, [10, 2, 3, 5, 6, 8, 9, 12])

The standard procedure for obtaining a scalar DE admitting a prescribed Lie algebra of symmetries is that of computing the differential invariants of its prolonged action, under some regularity hypotheses; the invariant DE is then described by the vanishing of an arbitrary function of such invariants. If the prolonged action is not regular, invariant DEs can be obtained by a careful study of the singular set of the aforementioned action.

We call *Lie remarkable* a DE which is completely characterized by its Lie algebra of point symmetries. In [7] we study Lie remarkable ODEs (both scalar and systems) that are associated with relevant Lie algebras of vector fields on $\mathbb{R}^k$. As the first step, we obtain scalar Lie remarkable ODEs by means of the local classification of primitive Lie algebras of vector fields on $\mathbb{R}^2$ (a list of such Lie algebras can be found in [**?**]). Note that they include the euclidean, affine, special conformal and projective Lie algebra of $\mathbb{R}^2$. Then we concentrate on the computations of Lie remarkable systems of ODEs. The ODEs that we obtain as our results are in part well-known, and in part new. Since known equations have interesting integrability properties, we expect that the new results could be of interest too.

All computations are performed through the use of the computer algebra package ReLie [11], a REDUCE program developed by one of us (F.O.).

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