On the geometry of Hamiltonian formalism for partial differential equations

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The integrability of Partial Differential Equations (PDEs) is quite often obtained from the existence of distinguished algebraic and/or geometric structures. More particularly, a feature of many (non-linear) integrable differential equations is the existence of an infinite sequence ('hierarchy') of commuting higher (or generalized) symmetries and/or conservation laws. Such sequences are often generated through differential operators.

In particular, we were attracted by a distinguished class of such differential operators, the third-order homogeneous Hamiltonian operators, also known as third-order Dubrovin-Novikov operator. Such operators admit non-trivial examples on systems of PDEs in $1+1$ independent variables. The most important example of PDE which admit such an operator is the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation in 3 components [2]:

$$u^1_t = \frac{1}{2}(u^2u^3 - u^1u^2 - u^1u^3)_x,$$
$$u^2_t = \frac{1}{2}(u^1u^3 - u^2u^1 - u^2u^3)_x,$$
$$u^3_t = \frac{1}{2}(u^1u^2 - u^3u^1 - u^3u^2)_x.$$

(see also [4]).

Using a characterization of such operators [7] we classified all of them under the action of of a group of symmetries which is closely related with the group of projective transformations of dependent variables [3]. The results have been exposed in several universities [8, 9, 10, 11].

In the above computations we were supported by symbolic software. P.H.M. Kersten and the symbolic software group of the University of Twente, and later R. Vitolo, developed the program CDIFF that runs in the symbolic environment REDUCE (which is now free software). This is one of the few publicly available programs which is able to make computations on integrability differential operators. The theory on which the software is based is exposed in a series of papers [5, 6]

Computations for multidimensional PDEs proved to be so hard to push symbolic software to its limits; this led, in cooperation with REDUCE developers to a general improvement of the program itself. R. Vitolo now is one of REDUCE developers, see `http://sourceforge.net/projects/reduce-algebra/`. More details on CDIFF, a user guide and many examples of use can be found here [12].

Together with J.S. Krasil'shchik and A.M. Verbovetsky, we plan to publish a scientific paper on CDIFF within 2014.

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10. M.V. Pavlov, *On the classification of third-order homogeneous Hamiltonian operators* invited talk at the Workshop on Geometry of PDEs and Integrability 14-18 October 2013, Teplice nad Becvou, Czech Republic.
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12. R. Vitolo, CDIFF: a REDUCE package for computations in geometry of PDEs, software, user guide and examples, version 3.5; freely available at the Geometry of the Differential Equations website `http://gdeq.org/Category:Software`. CDIFF is based on older software which is included in the official REDUCE distribution `http://sourceforge.net/projects/reduce-algebra/`.