Lie solvable enveloping algebras of characteristic two

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Let A be an associative algebra over a field $\mathbb{F}$. Then A can be regarded as a Lie algebra by means of the Lie bracket defined by $[x, y] = xy - yx$, for every $x, y \in A$. The algebra A is said to be Lie solvable if it is solvable as a Lie algebra.

Lie solvable algebras have been extensively studied over the years. There has been a special attention to group algebras. Let $\mathbb{F}G$ be the group algebra of the group G over a field $\mathbb{F}$. Recall that G is said to be p-abelian if $p > 0$ and G', the commutator subgroup of G, is a finite p-group. Moreover, in the zero characteristic case we say that G is 0-abelian if it is abelian. Passi, Passman and Sehgal in [15] proved that a group algebra $\mathbb{F}G$ is Lie solvable if and only if either $\mathrm{char}(\mathbb{F}) \neq 2$ and G is p-abelian or $\mathrm{char}(\mathbb{F}) = 2$ and G has a 2-abelian subgroup of index at most 2.

Restricted Lie algebras and p-groups enjoy similar properties and so it was of interest to find an analogue of Passi-Passman-Sehgal's result for restricted Lie algebras. Let L be a restricted Lie algebra over a field of positive characteristic p and denote by $u(L)$ the restricted (universal) enveloping algebra of L. Riley and Shalev in early 1990s proved that if $p \neq 2$ then $u(L)$ is Lie solvable if and only if L' (the derived subalgebra of L) is finite-dimensional and p-nilpotent. However, they left out the even characteristic case. The purpose of the present paper is to fill this gap, thereby completing the classification. Our main result shows that the analogue of group ring case in $p = 2$ fails for restricted Lie algebras and indeed, as we shall see below, the characterizations in $p = 2$ case are significantly different.

A polynomial identity (PI) is called non-matrix if it is not satisfied by the algebra $M_2(\mathbb{F})$ of 2 by 2 matrices over $\mathbb{F}$. Note that Lie solvability is a non-matrix PI provided that $\mathrm{char}(\mathbb{F}) \neq 2$. Indeed, if $\mathrm{char}(\mathbb{F}) = 2$ then $M_2(\mathbb{F})$ is Lie center-by-metabelian. The non-matrix varieties of algebras have been extensively studied, see for example [10], [12], [13], [20], and enveloping algebras have received special attention in this respect [3], [4], [23]. Using the standard PI-theory, like Posner's Theorem, one can deduce that if R is an associative algebra that satisfies a non-matrix PI over a field $\mathbb{F}$ of characteristic p then $[R, R]R$ is nil. If we further assume that R is Lie solvable and $p \neq 2$, then $[R, R]R$ is nil of bounded index (see [20]). Moreover, if we restrict ourselves to $R = u(L)$ then R satisfies a non-matrix PI if and only if $[R, R]R$ is nil of bounded index (see [23]). However, if $u(L)$ is Lie solvable and $p = 2$ then L' may not be even nil as we shall see below in our main result.

In order to state the main result, we recall a few definitions. A subset S of L is said to be p-nilpotent if there exists $m > 0$ such that $S^{[p]^m} = \{x^{[p]^m} \mid x \in S\} = 0$. We denote by $Z(L)$ the center of L. Following [7], we say that a restricted Lie algebra is strongly abelian if it is abelian and its power mapping is zero. In analogy with group rings, we say that a restricted subalgebra H of L is *p-abelian* if H' is finite-dimensional and p-nilpotent. For a subset X of L we denote by $\langle X \rangle_{\mathbb{F}}$ the vector subspace spanned by X. Our main result is the following:

Main Theorem. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic 2. Let $\bar{\mathbb{F}}$ be the algebraic closure of $\mathbb{F}$ and set $\mathfrak{L} = L \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Then $u(L)$ is Lie solvable if and only if $\mathfrak{L}$ has a finite-dimensional 2-nilpotent restricted ideal I such that $\bar{\mathfrak{L}} = \mathfrak{L}/I$ satisfies one of the following conditions:*

(i) *$\bar{\mathfrak{L}}$ has an abelian restricted ideal of codimension at most 1;*

(ii) *$\bar{\mathfrak{L}}$ is nilpotent of class 2 and $\dim \bar{\mathfrak{L}}/Z(\bar{\mathfrak{L}}) = 3$;*

(iii) *$\bar{\mathfrak{L}} = \langle x_1, x_2, y \rangle_{\bar{\mathbb{F}}} \oplus Z(\bar{\mathfrak{L}})$, where $[x_1, y] = x_1$, $[x_2, y] = x_2$, and $[x_1, x_2] \in Z(\bar{\mathfrak{L}})$;*

(iv) *$\bar{\mathfrak{L}} = \langle x, y \rangle_{\bar{\mathbb{F}}} \oplus H \oplus Z(\bar{\mathfrak{L}})$, where H is a strongly abelian finite-dimensional restricted subalgebra of $\bar{\mathfrak{L}}$ such that $[x, y] = x$, $[y, h] = h$, and $[x, h] \in Z(\bar{\mathfrak{L}})$ for every $h \in H$;*

(v) *$\bar{\mathfrak{L}} = \langle x, y \rangle_{\bar{\mathbb{F}}} \oplus H \oplus Z(\bar{\mathfrak{L}})$, where H is a finite-dimensional abelian subalgebra of $\bar{\mathfrak{L}}$ such that $[x, y] = x$, $[y, h] = h$, $[x, h] \in Z(\bar{\mathfrak{L}})$, and $[x, h]^{[2]} = h^{[2]}$, for every $h \in H$.*

We also show that the extension of the ground field is necessary in the statement of our main theorem. Furthermore, note that the cases (ii)-(v) can occur only when L' is finite-dimensional. In other words, if $u(L)$ is Lie solvable and L' has infinite dimension, then L has a 2-abelian restricted ideal of codimension at most 1.

In the last two decades there has been some interest on the derived length of Lie solvable group algebras and enveloping algebras (see [6], [22], [26], [27], [30], [31], [33]), and small characteristics have been considered separately, see for example [11], [24], [29]. It is also worth mentioning that besides the interest on their own, restricted enveloping algebras occur naturally in the study of graded group rings (see e.g. [18], [25]). For instance, by using this approach, in [25] Shalev showed that a graded group ring of a finitely generated group ring over a field of characteristic $p > 0$ satisfies a polynomial identity if and only if the pro-p completion of G has the structure of a p-adic Lie group.

Finally, let L be a Lie algebra over an arbitrary field $\mathbb{F}$ and let $U(L)$ denote the ordinary universal enveloping algebra of L. Necessary and sufficient conditions for $U(L)$ to satisfy a polynomial identity have been found in [1]. Moreover, it is known that if $\mathbb{F}$ has characteristic different from 2, then $U(L)$ is Lie solvable only when L is abelian (see [21, §6, Corollary 6.1]). This is no longer true in characteristic 2. As an application of our main theorem, in the concluding section a description of Lie solvable enveloping algebras in characteristic 2 will be obtained, thereby completing the characterization also in the ordinary case.

REFERENCES

1. Yu.A. Bahturin: *Identities in the universal envelopes of Lie algebras*, J. Austral. Math. Soc. **18** (1974), 19–27.
2. Yu.A. Bahturin, A.A. Mikhalev, V.M. Petrogradsky, M. V. Zaicev: *Infinite-dimensional Lie Superalgebras*. de Gruyter Expositions in Mathematics, Vol. 7, Walter de Gruyter & Co., Berlin, 1992.
3. J. Bergen, D.M. Riley, H. Usefi: *Lie superalgebras whose enveloping algebras satisfying a non-matrix polynomial identity*, Israel J. Math, to appear.
4. Y. Billig, D. Riley, V. Tasic: *Nonmatrix varieties and nil-generated algebras whose units satisfy a group identity*, J. Algebra **190** (1997), no. 1, 241–252.
5. D. Burde, M. Ceballos: *Abelian ideals of maximal dimension for solvable Lie algebras*, J. Lie Theory (2012), 741–756.
6. F. Catino, S. Siciliano, E. Spinelli: *Restricted enveloping algebras with minimal Lie derived length*, Algebr. Represent. Theory **13** (2010), 653–660.
7. G. Hochschild: *Cohomology of restricted Lie algebras*, Amer. J. Math. **76** (1954), 555–580.
8. G. Hochschild: *Representations of restricted Lie algebras of characteristic p*, Proc. Amer. Math. Soc. **5** (1954), 603–605.
9. N. Jacobson: *Lie Algebras*. Wiley Interscience, New York, 1962.
10. A.R. Kemer: *Ideal of Identities of Associative Algebras*, (Amer. Math. Soc., Providence, RI, 1991), Vol. 87.
11. B. Külshammer, R.K. Sharma: *Lie centrally metabelian group rings in characteristic 3*, J. Algebra **180** (1996), 111–120.
12. V.N. Latyshev: *Nonmatrix varieties of associative algebras*, *Mat. Zametki* **27** (1980), no. 1, 147–156.
13. S.P. Mishchenko, V.M. Petrogradsky, A. Regev: *Characterization of non-matrix varieties of associative algebras*, Israel J. Math., **182** (2011), 337-348.
14. S. Montgomery: *Hopf algebras and their actions on rings*, CMBS Regional Conference Series in Mathematics, 1993.
15. I.B.S. Passi, D.S. Passman, S.K. Sehgal: *Lie solvable group rings*, Canad. J. Math. **25** (1973), 748–757.
16. D.S. Passman: *Enveloping algebras satisfying a polynomial identity*, J. Algebra **134** (1990), 469–490.
17. V.M. Petrogradsky: *Existence of identities in the restricted enveloping algebra*, Math. Zametki **49** (1991), 84–93.
18. D.G. Quillen: *On the associated graded ring of a group ring*, J. Algebra **10** (1968), 411–418.
19. H. Radjavi, P. Rosenthal: *Simultaneous Triangularization*. Universitext, Springer-Verlag, 2000.
20. D.M. Riley: *PI-algebras generated by nilpotent elements of bounded index*, J. Algebra **192** (1997), no. 1, 1–13.
21. D.M. Riley, A. Shalev: *The Lie structure of enveloping algebras*, J. Algebra **162** (1993), 46–61.
22. D.M. Riley, V. Tasić: *Lie identities for Hopf algebras*, J. Pure Appl. Algebra **122** (1997), 127–134.
23. D.M. Riley, M.C. Wilson: Group algebras and enveloping algebras with nonmatrix and semigroup

identities, *Comm. Algebra* **27** (7) (1999), 3545–3556.
24. R. Rossmanith: *Lie centre-by-metabelian group algebras in even characteristic. I, II.*, Israel J. Math. **115** (2000), 51–75, 77–99.
25. A. Shalev: *Polynomial identities in graded group rings, restricted Lie algebras and p-adics analytic groups*, Trans. Amer. Math. Soc. **337**(1993), 451–462.
26. A. Shalev: *The derived length of Lie soluble group rings. I.*, J. Pure Appl. Algebra **78**(1992), 291–300.
27. A. Shalev: *The derived length of Lie soluble group rings. II.*, J. London Math. Soc. (2) **49**(1994), 93–99.
28. R.K. Sharma, J.B. Srivastava: *Lie solvable rings*, Proc. Amer. Math. Soc. **94** (1985), 1–8.
29. R.K. Sharma, J.B. Srivastava: *Lie centrally metabelian group rings in characteristic 3*, J. Algebra **180** (1996), 111–120.
30. S. Siciliano: *Lie derived lengths of restricted universal enveloping algebras*, Publ. Math. Debrecen **68** (2006), 503–513.
31. S. Siciliano: *On Lie solvable restricted enveloping algebras*, J. Algebra **314** (2007), 226–234.
32. M.B. Smirnov, A.E. Zalesski: *Associative rings satisfying the identity of Lie solvability*, Vestsi Akad. Navuk BSSR Ser. Fiz.-Mat. Navuk **123** (1982), 15–20.
33. E. Spinelli: *Group algebras with minimal Lie derived length*, J. Algebra **320** (2008), 1908–1913.
34. H. Strade: *Simple Lie algebras over fields of positive characteristic I. Structure theory.* Walter de Gruyter & Co., Berlin/New York, 2004.
35. H. Strade, R. Farnsteiner: *Modular Lie algebras and their representations.* Marcel Dekker, New York, 1988.
36. D.J. Winter: *Abstract Lie algebras.* MIT Press, Cambridge, Mass, 1972.