Lie identities on symmetric elements of restricted enveloping algebras

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Let A be an algebra with involution $*$ over a field $\mathbb{F}$. We denote by $A^+ := \{x \in A | \, x^* = x\}$ the set of symmetric elements of A under $*$ and by $A^- := \{x \in A | \, x^* = -x\}$ the set of skew-symmetric elements. A question of general interest is which properties of A^+ or A^- can be lifted to the whole algebra A. The history of this problem goes back to Herstein [12], [13], where he had conjectured that if the symmetric or skew-symmetric elements of a ring R satisfy a polynomial identity, then so does R. Notably this conjecture was proved by Amitsur in [1] and subsequently generalized by himself in [2].

Now consider the group algebra $\mathbb{F}G$ of a group G under the canonical involution induced by $g \mapsto g^{-1}$, for every $g \in G$. There has been an intensive investigation devoted to demonstrate the extent to which the symmetric or skew-symmetric elements of $\mathbb{F}G$ under the canonical involution determine the structure of the group algebra and there has been special attention on Lie identities. In particular, the characterization of groups G for which $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is Lie nilpotent was carried out by Giambruno, Sehgal, and Lee in [10], [11], [16]. Furthermore, if either $\mathbb{F}G^-$ or $\mathbb{F}G^+$ is bounded Lie Engel, and G is devoid of 2-elements, Lee in [17] showed that $\mathbb{F}G$ is bounded Lie Engel. He also classified remaining groups for which $\mathbb{F}G^+$ is bounded Lie Engel. The Lie solvable case was considered in [19], [20], however a complete answer to this case seems still under way.

Now, let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$ and let $u(L)$ be the restricted enveloping algebra of L. We denote by $\top$ the *principal involution* of $u(L)$, that is, the unique $\mathbb{F}$-antiautomorphism of $u(L)$ such that $x^\top = -x$ for every x in L. We recall that $\top$ is just the antipode of the $\mathbb{F}$-Hopf algebra $u(L)$.

Recently, the first author in [24] established the conditions under which $u(L)^-$ is Lie solvable, Lie nilpotent or bounded Lie Engel.

In this paper we consider the symmetric case. Unlike the skew-symmetric case it is not clear a priori that, for example, when $u(L)^+$ is Lie nilpotent then L is nilpotent. The symmetric elements do not form a Lie subalgebra of $u(L)$ in general (but they do form a Jordan subalgebra under the Jordan bracket $x \circ y = \frac{1}{2}(xy + yx)$). However, despite the group ring case we present a complete answer and yet the proofs are different and more involved than the skew-symmetric case. Before stating the main results we recall the following definitions. An element x of L is p-nilpotent if $x^{[p]^m} = 0$ for some $m \geq 1$; a subset S of L is p-nilpotent if $S^{[p]^m} = \{x^{[p]^m} | \, x \in S\} = 0$ for some $m \geq 1$. The derived subalgebra of L is denoted by L'.

Theorem 1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

1) *$u(L)^+$ is bounded Lie Engel;*

2) *$u(L)$ is bounded Lie Engel;*

3) *L is nilpotent, L' is p-nilpotent, and L contains a restricted ideal I such that L/I and I' are finite-dimensional.*

Theorem 2. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

1) *$u(L)^+$ is Lie nilpotent;*

2) *$u(L)$ is Lie nilpotent;*

3) *L is nilpotent and L' is finite-dimensional and p-nilpotent.*

Theorem 3. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 2$. Then the following conditions are equivalent:*

1) *$u(L)^+$ is Lie solvable;*
2) *$u(L)$ is Lie solvable;*
3) *L' is finite-dimensional and p-nilpotent.*

The equivalence of 2) and 3) in Theorems 1.1-1.3 was established in [23]. Our main contribution is to prove that 2) implies 3) in each of these theorems. In combination with [24], we can conclude that, in odd characteristic, if either $u(L)^+$ or $u(L)^-$ is Lie solvable (respectively, bounded Lie Engel or Lie nilpotent) then so is the whole algebra $u(L)$. Such conclusions are no longer true in characteristic 2.

Finally, let L be an arbitrary Lie algebra over a field of characteristic different from 2 and denote by $U(L)$ the ordinary enveloping algebra of L. A further consequence of our main results is that $U(L)^+$ (under the principal involution) is Lie solvable or bounded Lie Engel only when L is abelian.

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