Outer restricted derivations of nilpotent restricted Lie algebras

Jörg Feldvoss, [a], Salvatore Siciliano, [b], Thomas Weigel, [c]

[a]Department of Mathematics and Statistics, University of South Alabama, Mobile, AL 366880002, USA

[b]Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Via Provinciale Lecce–Arnesano, 73100–Lecce, Italy

[c]Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca, U5-3067, Via R. Cozzi 53, I-20125 Milano, Italy

In 1966 W. Gaschütz proved the following celebrated result:

Theorem 1. (W. Gaschütz [3]) *Every finite p-group of order $> p$ has an outer automorphism of p-power order.*

Since every finite nilpotent group is a direct product of its Sylow p-subgroups, the outer automorphism group of a finite nilpotent group is a direct product of the outer automorphism groups of its Sylow p-subgroups. Therefore it is a direct consequence of Gaschütz's theorem that every finite nilpotent group of order greater than 2 has an outer automorphism. This answers a question raised by E. Schenkman and F. Haimo in the affirmative (see [9]).

Since groups and Lie algebras often have structural properties in common, it seems rather natural to ask whether an analogue of Gaschutz's theorem holds in the setting of ordinary or restricted Lie algebras. In the case of ordinary Lie algebras a stronger version of such an analogue is already known and was established by S. Tôgô around the same time as Gaschütz proved his result.

Theorem 2. (S. Tôgô [11, Corollary 1]) *Every nilpotent Lie algebra of finite dimension > 1 over an arbitrary field has an outer derivation whose square is zero.*

In fact, Tôgô's result is more general (see [11, Theorem 1]) and is a refinement of a theorem of E. Schenkman that establishes the existence of outer derivations for non-zero finite-dimensional nilpotent Lie algebras (see [9, Theorem 4]). Much later the first author proved a restricted analogue of Schenkman's result for p-unipotent restricted Lie algebras (see [2, Corollary 5.2]). (Here we follow [1, Section I.4, Exercise 23, p. 97] by calling a restricted Lie algebra $(L, [p])$ p-unipotent if for every $x \in L$ there exists some positive integer n such that $x^{[p]^n} = 0$.) We recall that a derivation D of a restricted Lie algebra L is called restricted if $D(x^{[p]}) = (\mathrm{ad}_L x)^{p-1}(D(x))$ for every $x \in L$ (see [6, , Section 4, p. 21]) and the set of all restricted derivations of L is denoted by $\mathrm{Der}_p(L)$. Observe that $\mathrm{Der}_p(L)$ is a restricted Lie algebra (see [6, Theorem 4]) and that every inner derivation of L is restricted.

In this paper we prove that every finite-dimensional nilpotent restricted Lie algebra L over a field $\mathbb{F}$ of characteristic $p > 0$ has an outer restricted derivation whose square is zero unless L is a torus or $\dim_{\mathbb{F}} L = 1$ or $p = 2$ and L is isomorphic to the three-dimensional Heisenberg algebra $\mathfrak{h}_1(\mathbb{F})$ over $\mathbb{F}$ as an ordinary Lie algebra. Indeed, in the later three cases every nilpotent restricted derivation is inner. As a consequence we also obtain a generalization of [2, Corollary 5.2] to nontoral nilpotent restricted Lie algebras which is the full analogue of Schenkman's result for nilpotent restricted Lie algebras.

REFERENCES

1. N. Bourbaki: *Lie Groups and Lie Algebras.* Chapters 1–3, Springer-Verlag, Berlin/Heidelberg/New York/London/Paris/Tokyo, 1989.
2. J. Feldvoss: *On the cohomology of restricted Lie algebras*, Comm. Algebra **19** (1991), 2865–2906.
3. W. Gaschütz: *Nichtabelsche p-Gruppen besitzen äussere p–Automorphismen*, J. Algebra **4** (1966), 1–2.
4. G. Hochschild: *Cohomology of restricted Lie algebras*, Amer. J. Math. **76** (1954), 555–580.
5. G. Hochschild: *Representations of restricted Lie algebras of characteristic p*, Proc. Amer. Math. Soc. **5** (1954), 603–605.
6. N. Jacobson: *Restricted Lie algebras of characteristic p*, Trans. Amer. Math. Soc. **50** (1941), 15–25.

7. N. Jacobson: *A note on automorphisms and derivations of Lie algebras*, Proc. Amer. Math. Soc. **6** (1955), 281–283.
8. Jacobson, N.: *Lie algebras.* Wiley Interscience, New York, 1962.
9. E. Schenkman: *The existence of outer automorphisms of some nilpotent groups of class 2*, Proc. Amer. Math. Soc **6** (1965), 6–11.
10. H. Strade, R. Farnsteiner: *Modular Lie algebras and their representations.* Marcel Dekker, New York, 1988.
11. S. Tôgô: *Outer derivations of Lie algebras*, Trans. Amer. Math. Soc. **128** (1967), 264–276.