Split abelian chief factors and first degree cohomology for Lie algebras

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W. Gaschütz proved the "only if"-part of the following cohomological vanishing theorem for finite p-solvable groups (see [22, Lemma 1]), and the converse is due to U. Stammbach [22, Theorem A]. Here and in the following p is an arbitrary prime number, and G is a finite group whose order is divisible by p. Moreover, let $\mathbb{F}_p[G]$ denote the group algebra of G over the field $\mathbb{F}_p$ with p elements, and let $C_G(M) := \{g \in G \mid g \cdot m = m \text{ for every } m \in M\}$ denote the *centralizer* of a (unital left) $\mathbb{F}_p[G]$-module M in G.

Theorem 1. *A finite group G is p-solvable if, and only if, $H^1(G/C_G(S), S) = 0$ for every irreducible $\mathbb{F}_p[G]$-module S.*

Let S be an irreducible $\mathbb{F}_p[G]$-module. Then $[G : S]_{\mathrm{split}}$ denotes the number of *p-elementary abelian chief factors* G_j/G_{j-1} $(1 \leq j \leq n)$ of a given chief series $\{1\} = G_0 \subset G_1 \subset \cdots \subset G_n = G$ that are isomorphic to S as $\mathbb{F}_p[G]$-modules and for which the exact sequence $\{1\} \to G_j/G_{j-1} \to G/G_{j-1} \to G/G_j \to \{1\}$ splits in the category of groups. According to the main result of [23], $[G : S]_{\mathrm{split}}$ is independent of the choice of the chief series of G.

W. Gaschütz also proved the "only if"-part of the following result on split (or complementable) p-chief factors of finite p-solvable groups (see [17, Theorem VII.15.5]). The converse of Gaschütz' theorem is due to U. Stammbach [23, Corollary 1]), but in an equivalent form it was already proved earlier by W. Willems [25, Theorem 3.9].

Theorem 2. *A finite group G is p-solvable if, and only if, $\dim_{\mathbb{F}_p} H^1(G, S) = \dim_{\mathbb{F}_p} \mathrm{End}_{\mathbb{F}_p[G]}(S) \cdot [G : S]_{\mathrm{split}}$ holds for every irreducible $\mathbb{F}_p[G]$-module S.*

The main goal of this paper is to investigate whether analogues of Theorem 1 and Theorem 2 hold in the context of ordinary Lie algebras. Some time ago in an unpublished manuscript the first author of this paper has obtained analogues of these results for restricted Lie algebras (see [11, Remark after Theorem 7]). The crucial point in the argument is an analogue of Theorem 1 for ordinary Lie algebras (see [12, Proposition 1]). The proof in [12] was a modification of the proof of a similar characterization of supersolvable Lie algebras due to D. W. Barnes (see [2, Theorem 4]). It turns out that there is a gap in both these proofs which will be fixed in the present paper. This is achieved by applying Shapiro's lemma for truncated coinduced modules (see [5, Theorem in §5 and Corollary 1 in §3] or [8, Theorem 2.1 and Theorem 1.4]) in order to establish a refinement of a non-vanishing theorem of G. Seligman for the first degree cohomology of a non-solvable finite-dimensional Lie algebra in prime characteristic (see [21, p. 102]). In this regard our approach is similar to Stammbach's proof of the "if"-part of Theorem 1 (see the proof of [22, Lemma 2]).

We begin with a Lie-theoretic analogue of the main result of [23]. This result is already contained in the first author's unpublished manuscript (see [11, Lemma 5]), but the proof there implicitly uses that the multiplicity of split abelian chief factors isomorphic to a given irreducible module is independent of the chief series. The proof given here follows the argument used in the proof of [11, Lemma 5], but also deals with a case not considered in [11]. As a consequence of our result and Barnes' cohomological vanishing theorem for solvable Lie algebras (see [1, Theorem 2]), one obtains the Lie-theoretic analogue of Gaschütz' theorem on split p-chief factors. In the third section we show that for fields of characteristic zero the analogues of the conditions in Theorem 1 and Theorem 2 are always satisfied. Thus, as in the group case, only modular solvable Lie algebras can be characterized by these properties. This will be the main goal of the fourth section. In the final section we apply the results obtained in Sections 2 and 4 to restricted Lie algebras. The equivalence (i)$\Longleftrightarrow$(viii) in the last theorem of that section is an analogue of

Willems' module-theoretic characterization of p-solvable groups (see [25, Theorem 3.9]) for restricted Lie algebras. As a by-product we establish several results on the second Loewy layer of the projective cover of the trivial irreducible module. Most of the results in Section 5 were already contained in [11, Section 4], but have never been published before.

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