Heat semigroup and geometric measure theory

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Since its first definition given by E. de Giorgi in 1954 in [1], the notion of *variation* of a function of several real variables (which is called *perimeter* in the relevant case of characteristic functions [1]) appeared to be closely related to the short-time behaviour of the heat semigroup. If $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is in $L^1(\Omega)$ then its variation is defined as

$$|Du|(\Omega) = \sup\Big\{ \int_\Omega u \operatorname{div} \phi \; dx : \; \phi \in C^1_c(\Omega, \mathbb{R}^n), \; \|\phi\|_\infty \le 1 \Big\} \tag{1}$$

and the condition $|Du|(\Omega) < +\infty$ is equivalent to saying that the distributional gradient of u is an $\mathbb{R}^n$-valued Radon measure. In this case, we say that u is of *bounded variation*, $u \in BV(\Omega)$ for short, see [2]. Let us consider now the case $\Omega = \mathbb{R}^n$. Given $u \in L^1(\mathbb{R}^n)$, denote by W_t the *heat semigroup* in $\mathbb{R}^n$ given by

$$W_t u(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} u(y) p(t, x, y) \, dy, \tag{2}$$

with integral kernel

$$p(t, x, y) = \exp\Big\{ \frac{|x - y|^2}{4t} \Big\}. \tag{3}$$

The function $w(x, t) = W_t u(x)$ is the solution of the Cauchy problem in $\mathbb{R}^n \times (0, +\infty)$

$$w_t = \Delta w, \qquad w(x, 0) = u(x).$$

Then, the equality

$$|Du|(\mathbb{R}^n) = \lim_{t \to 0} \int_{\mathbb{R}^n} |\nabla W_t u(x)| dx \tag{4}$$

holds, in the sense that the right hand side is finite if and only if $u \in BV(\mathbb{R}^n)$. Further connections between the short-time behaviour of the heat (or related) semigroups have been studied thus far and some recent results are briefly summarised in my 2012 annual report. Here I point out some related new results that have been achieved in 2013.

1. Relation (4), with M in place of $\mathbb{R}^n$, was known in Riemannian manifolds M with Ricci curvature bounded below, see [3], [4]. In [5] the above result is generalised to a class of Riemannian manifolds with unbounded Ricci curvature. In order to give a precise statement, let me recall the that the minimal positive heat kernel $p(t, x, y)$ on M can be defined as the pointwise minimal function

$$p(\cdot, \cdot, \cdot) : (0, \infty) \times M \times M \longrightarrow (0, \infty)$$

with the property that for all fixed $y \in M$, the function $p(\cdot, \cdot, y)$ is a classic solution of

$$\partial_t w(x, t) = \frac{1}{2} \Delta w(x, t), \quad \lim_{t \to 0+} w(\cdot, t) = \delta_y,$$

where of course classic solution means that w is C^1 in the time variable and C^2 in the space variable. It follows from parabolic regularity that $p(t, x, y)$ is smooth in (t, x, y), and furthermore $p(t, \cdot, \cdot)$ is the unique continuous version of the integral kernel of the heat semigroup on M, as in (2), (3). Now we can say that a Borel function $f : M \to \mathbb{C}$ is in the *Kato class* of M if

$$\lim_{t \to 0+} \sup_{x \in M} \int_0^t \int_M p(s, x, y) |f(y)| \mathrm{vol}(dy) \, ds = 0.$$

[1]By characteristic function of the set E we mean $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$.

It is easily seen that bounded functions are strictly contained in the Kato class. In [5] it is proved that (4) holds, with M in place of $\mathbb{R}^n$ and W_t the minimal heat semigroup in M, defined as in (2) through the minimal heat kernel on M, provided that the negative part Ric_- of the Ricci curvature of M belongs to the Kato class. Therefore, some (not too wild) unboundedness of Ric_- is allowed.

2. Functions of bounded variation can be defined also in an infinite dimensional separable Banach space X endowed with a Gaussian measure $\gamma = \mathcal{N}(0, Q)$ (centred normal distribution with covariance Q) and a differential strucure coming from the *Malliavin calculus.* The point of view is related to stochastic processes that are solutions of a particular stochastic differential equations and give rise to an *Orstein-Uhlenbeck* semigroup. Problems analogous to (4) can be raised in this context, known as *Wiener space.* Some recent results in this direction are presented in my 2012 annual report as well, and an extensive report, based on some lectures delivered by M. Novaga and myself, is presented in [7]. In this same context, equality (4) has been shown in [6] when a convex set Ω in the Wiener space is considered, and the heat semigroup W_t is replaced by the Ornstein-Uhlenbeck semigroup with suitable boundary conditions.

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