Magnetic Domains in the Skyrme - Faddeev model

Luigi Martina [1] [2]

[1]Dipartimento di Fisica, Università del Salento, Italy

[2]Istituto Nazionale di Fisica Nucleare sez. di Lecce, Italy

1. Introduction

The present contribution is mainly based on the developments of the results reported in the recent paper [1], where several reductions of the the 4-dimensional relativistic Skyrme-Faddeev model[2] [3] were solved implicitly or in terms of special functions, the action functional being

$$\begin{aligned}\mathcal{L} \;\; = \;\; & \frac{1}{32\pi^2}\,\Big(\partial_\mu\phi\cdot\partial^\mu\phi - \kappa\,(1-\phi\cdot\phi) \\ & -\frac{\lambda}{4}\,(\partial_\mu\phi\times\partial_\nu\phi)\cdot(\partial^\mu\phi\times\partial^\nu\phi)\Big)\,,\end{aligned}$$

where $\lambda > 0$ is a scaling parameter, determining the breaking of the conformal symmetry. The Lagrangian multiplier κ implements the unimodularity $\phi\cdot\phi = 1$ condition, which allows us to introduce the polar representation

$$\phi = (\sin w \cos u, \sin w \sin u, \cos w)\,,$$

where w and u are suitable functions. The Euler–Lagrange equations read

$$\begin{gathered}\partial_\mu w^\mu = \tfrac{1}{2}\sin(2w)u_\nu u^\nu + \\ \tfrac{\lambda}{2}\sin w\; u_\nu\,\partial_\mu[\sin w(w^\mu u^\nu - w^\nu u^\mu)], \\ w_\mu u^\mu \sin(2w) + \sin^2 w\;[\partial_\mu u^\mu + \\ \tfrac{\lambda}{2} w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu)] = 0 \qquad .\end{gathered}$$

The asymmetric structure of the equations, with respect the derivatives of u and w, suggests to introduce supplementary constraints. Actually, reductions will lead to extended infinite energy solutions in the 3-dimensional space, which are rarely considered in the current literature on the subject, that being focused on the hopfion solutions [2] - [5], mainly by numerical integration of the equations of motion. The present approach is motivated by: i) find analytical solutions to the system by exploiting its hidden symmetries, ii) give a suitable mathematical description of the of the extended phases observed in experimentsin ferromagnets and multiferroincs [6,7], iii) find a method in order to extract as as we can from the general model special subsectors completely integrables. From our view point, a strong connection with the d'Alembert-Eikonal system has been obtained. This observation leads to find large classes of solutions of the Cauchy problem. Here special classes are described and interpreted as magnetic domains for the considered model. Certain exact analytic periodic spin waves solutions are given in terms of elliptic integrals of third kind, obtaining quasi-periodic solutions. The Lagrangian can be averaged by the Whitham method, describing the modulation of periodic waves in terms of a quasilinear system of partial derivatives of the first order.

2. Domain-Walls

First we assume $w = const$, which reduces the d'Alembert - homogeneous Eikonal system, i.e.

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = 0.$$

This overdetermined system was investigated in many papers [8,9] and its general solution is implicitly given by

$$\begin{gathered}G\left(u, A_\mu\left(u\right)x^\mu, B_\mu\left(u\right)x^\mu\right) = 0, \\ A_\mu A^\mu = B_\mu B^\mu = A_\mu B^\mu = 0,\end{gathered}$$

with G, A_μ and B_μ arbitrary real regular functions. The process to provide explicit form for u may suggest that multi-valued functions appear. In fact, for sake of simplicity, restricting ourselves to 2D, imposing $B_\mu \equiv 0$ and for arbitrary f and F one can write the solution in the form

$$z = \arccos\left(\frac{x^1}{r}\right), \quad y = \cos\left(f\left(u\right) - z\right),$$

$$\arccos y = f\left(F\left(x^0 - ry\right)\right) - z, \quad r = \sqrt{\left(x^i\right)^2}.$$

However, such a reduction has few of interest, because of: i) asymptotics in time lead to the same boundary value and the solution becomes uniform, ii) the energy density is vanishing everywhere. So, looking for less drastic reductions like

$$w_\mu u^\mu = 0, \qquad\qquad u_\nu u^\nu = \alpha \quad ,$$

the original system reduces to the separable set of equations

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha, \quad w_\mu u^\mu = 0,$$
$$\partial_\mu w^\mu = \tfrac{\alpha}{2} \frac{\sin(2w)}{1-\frac{\lambda\alpha}{2}\sin^2 w}(1+\tfrac{\lambda}{2}w^\mu w_\mu).$$

The general solution of the d'Alembert-Eikonal (sub)-system is given [9] in the implicit form for the u and the auxiliary τ by

$$\begin{aligned} u &= A_\mu(\tau)\,x^\mu + R_1(\tau), \\ & \quad B_\mu(\tau)\,x^\mu + R_2(\tau) = 0, \\ A_\mu A^\mu &= \alpha, \quad A_\mu B^\mu = A'_\mu B^\mu = B_\mu B^\mu = 0, \end{aligned}$$

with arbitrary differentiable functions A_μ, B_μ, satisfying the constraints above, and functions R_i. If $\alpha < 0$, setting $\alpha = -\eta^2$, the solution becomes

$$u = x^k A_k(\tau) + A_0(\tau),$$
$$x^0 = x^k B_k(\tau) + B_0(\tau),$$
$$(A_i) = \eta \hat{\mathbf{A}}\left(f(\tau), g(\tau)\right)\ \left(\hat{\mathbf{A}}^2 = 1\right),$$
$$(B_i) = \hat{\mathbf{B}} = \pm\frac{\hat{\mathbf{A}}\times\hat{\mathbf{A}}'}{|\hat{\mathbf{A}}\times\hat{\mathbf{A}}'|}\left[f(\tau), g(\tau)\right],$$

where $f(\tau)$, $g(\tau)$, $A_0(\tau)$ and $B_0(\tau)$ are arbitrary differentiable functions. If $B_0(\tau)$ is not an injective continuous function, caustics and singularities of the wave front may appear. We know that in general the singularities of the wave fronts are classified by the Coxeter groups [10].

A different reduction, obtained by setting to zero the coefficients of all functions of w, leads to the overdetermined quasilinear system (named the reduced Skyrme–Faddeev system)

$$\begin{aligned} &\partial_\mu w^\mu = 0, && w_\mu w^\mu = -\epsilon^2, \\ &u_\mu w^\mu = 0, && a_\mu w^\mu = 0 \quad \text{with} \quad a = u^\nu u_\nu, \end{aligned}$$

where $\epsilon^2 = \frac{2}{\lambda}$ for notational clarity. After the change $u \to w$ and $\alpha \to -\epsilon^2$, the general solution of the d'Alembert-Eikonal system was discussed above. Cross differentiations of those equations and systematic substitutions of the x^0-derivatives lead to a set of compatibility conditions, which, as usual, the Monge-Ampére equation Det $[w_{ij}] = 0$, and involved quadratic constraints for the derivatives u_k. They can be simplified and possibly solved, if one can find a first order linear system of the form

$$u_0 = Au_1, \quad u_2 = Bu_1, \quad u_3 = Cu_1,$$

where the functions A, B, C depend w_m and $\partial_n w_m$ only, in analogy with the method of the hydrodynamic reductions [11,12], involving a finite number of Riemann invariants. This can be certainty done in two space dimensions, where one can prove that a solution for u is given by

$u = F[w_1, w_2]$, F arbitrary real differentiable and the quantity a vanishes.

In three space dimensions, a similar analysis is much more complicated, but the long expressions for A, B, C are intractable. On the other hand, it is easy to prove that a class of solutions for u is given by

$$u = F[w_1, w_2, w_3],$$

being F an arbitrary differentiable real function, constrained by the equation for a, which is not longer an identity.

A direct substitution of the above found solution for w into the orthogonality condition for $u_\mu\, w^\mu = 0$ leads to a linear PDE for u. By the methods of characteristics it is equivalent to a constant coefficient linear 3-ODE-system nilpotent of order 2, because of the peculiar symmetry of the constraints for A_μ and B_μ. Thus, from its solution one can extract two integrals of motion, in terms of which express the solution of the PDE for u. Then the general solution for the reduced Skyrme - Faddeev system is determined by five arbitrary functions: four for w and one for u.

3. Conclusions

We have found several reductions of the Skyrme-Faddeev model, showing the strong connection with the d'Alembert-Eikonal system, which allows to find large classes of solutions depending on the initial data for w and u. In particular, for the reduced Skyrme-Faddeev system we have shown that the u sub-system can be formulated as a first order system of linear pde's, which can be solved by the "method of hydrodynamic reductions ". Only, we remark that the constraint is a further restriction on the subsector of the solution space described in [13]. Finally, we have found [1] exact analytic periodic spin waves for the Skyrme-Faddeev model, determined in terms of elliptic integrals of third kind. Assuming that there exist such periodic dispersive wave-train solutions, we have shown that the Lagrangian can be averaged by the Whitham method. This provides a Lagrangian for a set of parameters, describing the modulation of periodic waves in terms of a quasilinear system of partial derivatives of the first order

Acknowledgments

The work was partially supported by the PRIN "Geometric Methods in the Theory of Nonlinear Waves and their Applications" of the Italian MIUR and by the INFN - Sezione of Lecce under the project LE41.

REFERENCES

1. L. Martina, M.V. Pavlov and S. Zykov: Waves in the Skyrme–Faddeev model and Integrable reductions, J. Phys. A: Math. Theor. 46 275201 (2013)
2. L. D. Faddeev: "Quantization of solitons", Princeton preprint IAS-75-QS70 (1975) ; L. D. Faddeev: "Some comments on the many-dimensional solitons" , Lett. Math. Phys. **1** n. 4, (1976) 289-293
3. Kundu A and Rybakov Y P 1982 Closed-vortex-type solitons with Hopf index *J. Phys. A: Math. Gen.* **15** 269
4. Faddeev L D and Niemi A J 1997 Stable knot-like structures in classical field theory *Nature* **387** 58
5. Sutcliffe P 2007 Knots in the Skyrme–Faddeev mode *Proc. R. Soc.* **A 463** 3001
6. Yu X Z *et al.* 2010 Real-space observation of a two-dimensional skyrmion crystal *Nature* 465, 901-904.
7. Romming N *et al.* 2013 Writing and Deleting Single Magnetic Skyrmions *Science* **341** 636-639
8. Cieciura G and Grundland A M 1984 A certain class of solutions of the nonlinear wave equation *J. Math. Phys.* **2** 3460-3469
9. Fushchich V I, Zhdanov R Z and Revenko I V 1991 General solutions of the nonlinear wave equation and of the eikonal equation *Ukr. Mat. Z.* **43** 1471-1487; Zhdanov R Z, Revenko I V and Fushchich V I 1995 On the general solution of the d'Alembert equation with a nonlinear eikonal constraint and its applications *J. Math. Phys.* **36** 7109
10. Arnold V I 1990 *Singularities of caustics and wave fronts* (Dordrecht, Kluwer Acad. Pub.)
11. Ferapontov E V and Pavlov M V 2003 Hydrodynamic reductions of the heavenly equation *Class. Quantum Grav.* **20** 2429-2441.
12. Ferapontov E V, Khusnutdinova K R and Tsarev S P 2006 On a class of three-dimensional integrable Lagrangians *Comm. Math. Phys.* **261** no. 1, 225–243
13. Alvarez O, Ferreira L A and Sanchez-Guillen J 2009 Integrable theories and loop spaces: fundamentals, applications and new developments *Int.J.Mod.Phys.* **A24** 1825-1888.