A non–Boltzmannian behaviour of the energy distribution for quasi stationary regimes of the Fermi–Pasta–Ulam β system

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One of the central results of classical statistical mechanics is the emergence of the Boltzmann weight, i.e. of the exponential decay of probabilities in complex systems, when we deal with independent identically distributed random variables. The standard theoretical framework justifying this phenomenon is the theory of large deviations. Since the seminal works of Cramer in the 1930's, Gärtner, Donsker and Varadhan in the 1970's, and Ellis in the 1980's, it is crucial in the description of stochastic processes.

This theory reposes essentially on two results. The first one, the Gärtner–Ellis theorem, establishes under which conditions the asymptotic behaviour of a random variable A is of the form $e^{-nr(x)}$, where the rate function $r(x)$ has the physical meaning of an entropy. The second one is the Cramer theorem, which ensures the emergence of the same asymptotic behaviour for a sample mean of independent and identically distributed random variables.

However, no generalization of this theory is presently known for the case of correlated variables, nor a generalization of the Boltzmann weight rigorously deduced from first principles.

At the same time, since the historical book by Gibbs on Elementary Principles in Statistical Mechanics, it was emphasized that the Boltzmann–Gibbs (BG) theory is unapplicable in several cases, for instance for systems involving long–range interactions. This fact reflects in many aspects, as the divergence of the associated partition functions.

In the last decades, a new theoretical framework, called nonextensive statistical mechanics, has appeared for describing the thermostatistics of systems typically exhibiting long–range correlations, (asymptotic) scale invariance, multifractality. The nonextensive approach generalizes the classical Boltzmann–Gibbs (BG) statistics in the sense that it applies to non ergodic, e.g. weakly chaotic, systems. Other relevant applications have been found, for example, in economics, linguistics, biosciences, social sciences, self–organized criticality.

In the paper [1], we show that a q–exponential weight could be a natural candidate for a generalization of the Boltzmann weight when correlations among random variables describing physical observables play a crucial role.

In this spirit, we propose a study of the energy distribution of the modal solutions of the Fermi–Pasta–Ulam (FPU) β–system, which represents one of most paradigmatic examples of classical Hamiltonian system. It admits an interesting class of solutions: the *one–mode solutions* (OMS). These are exact solutions, usually referred to by means of the values of the integer mode number $n = \frac{N}{4}$, $\frac{N}{3}$, $\frac{N}{2}$, $\frac{2}{3}N$, $\frac{3}{4}N$, where N is the number of the particles of the system.

In previous papers [2–6] we have studied both numerically and analytically the stability of the OMS. If E is the energy of the system and $\epsilon = E/N$ the energy density, these exact non linear solutions are characterized by stability threshold values of ϵ which, for high values of number of particles N, scale as $1/N^2$. These threshold values mark the transition of the system from a stable towards a recurrent and then a chaotic behaviour. Furthermore, by using a suitable global indicator, we have discovered that, by increasing ϵ, there is a second threshold for the modes $n = N/3$ and $n = N/4$ after which the two modes become stable again. Then it appears interesting to analyze, in the region of instability, the statistical distributions of quantities that are constant when the system is stable.

A connection between the weakly chaotic dynamics of the FPU model and nonextensive statistical mechanics was first established in [5,7]. In this work we perform a statistical analysis of the energy distributions of the $N/4$, $N/3$ and $N/2$ modes when they are unstable, namely for values of ϵ between the two thresholds for the modes $N/3$ and $N/4$ [6] and for values of ϵ greater than the single threshold for the $N/2$ mode. Our main result is that *these energy distributions deviate significantly from the classical Boltzmann weight.* Indeed, we obtain striking numerical evidence that a *q–exponential weight* should instead be used.

At the best of our knowledge, this is the first example of a physical system for which this behaviour has been shown numerically.

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