Improved lower bounds on the price of stability of undirected network design games

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Network design is among the most well-studied problems in the combinatorial optimization literature. A natural definition is as follows. We are given a graph consisting of a set of nodes and edges among them representing potential links. Each edge has an associated cost which corresponds to the cost for establishing the corresponding link. We are also given connectivity requirements as pairs of source-destination nodes. The objective is to compute a subgraph of the original graph of minimum total cost that satisfies the connectivity requirements. In other words, we seek to establish a network that satisfies the connectivity requirements at the minimum cost. This optimization problem is known as Minimum Steiner Forest and generalizes well-studied problems such as the Minimum Spanning Tree and Minimum Steiner Tree.

In this paper, we consider a game-theoretic variant of network design that was first considered in [2]. Instead of considering the connectivity requirements as a global goal, we assume that each connectivity requirement is desirable by a different player. The players participate in a non-cooperative game; each of them selects as her strategy a path from her source to the destination and is charged for part of the cost of the edges she uses. According to the fair cost sharing scheme we consider in the current paper, the cost of an edge is shared equally among the players using the edge. The *social cost* of an *assignment* (i.e., a snapshot of players' strategies) is the cost of the edges contained in at least one path. An optimal assignment would contain a set of edges of minimum cost so that the connectivity requirements of the players are satisfied. Unfortunately, this does not necessarily mean that all players are satisfied with this assignment since a player may have an incentive to deviate from her path to another one so that her individual cost is smaller. Eventually, the players will reach a set of strategies (and a corresponding network) that satisfies their connectivity requirements and in which no player has any incentive to deviate to another path; such outcomes are known as *Nash equilibria*. Interestingly, even though the optimal solution is always a forest, Nash equilibria may contain cycles.

The non-optimality of the outcomes of *network design games* (which is typical when selfish behavior comes into play) leads to the following question that has been a main line of research in Algorithmic Game Theory: How is the system performance affected by selfish behavior? The notion of the *price of anarchy* (introduced in the seminal paper of Koutsoupias and Papadimitriou [8]) quantifies the deterioration of performance. In general terms, it is defined as the ratio of the social cost of the worst possible Nash equilibrium over the optimal cost. Hence, it is pessimistic in nature and (as its name suggests) provides a worst-case guarantee for conditions of total anarchy. Instead, the notion of the *price of stability* that was introduced in the paper of Anshelevich *et al.* [2] is optimistic in nature. It is defined as the ratio of the social cost of the best equilibrium over the optimal cost and essentially asks: What is the best one can hope for the system performance given that the players are selfish?

The aim of the current paper is to determine better lower bounds on the price of stability for network design games in an attempt to understand the effect of selfishness on the efficiency of outcomes in such games. We usually refer to network design games as multi-source network design games in order to capture the most general case in which players may have different sources. An interesting variant is when each player wishes to connect a particular common node, which we call the *root*, with her destination node; we refer to such network design games as multicast games. An interesting special case of multicast games is the class of broadcast games: in such games, there is a player for each non-root node of the network that has this node as her destination.

The existence of Nash equilibria in network design games is guaranteed by a potential function argument. Rosenthal [10] defined a potential function over all assignments of a network design game so that the difference in the potential of two assignments that differ in the strategy of a single player equals the difference of the cost of that player in these assignments; hence, an assignment that locally minimizes the potential function is a Nash equilibrium. So, the price of stability is well-defined in network design games. Anshelevich *et al.* [2] considered network design games in directed graphs and proved that the price of stability is at most H_n. Their proof considers a Nash equilibrium that can be reached from an optimal assignment when the players make arbitrary selfish moves. The main argument used is that the potential of the Nash equilibrium is strictly smaller than that of the optimal assignment and the proof follows due to the fact that the potential function of Rosenthal approximates the social cost of an assignment within a factor of at most H_n. This approach suggests a general technique for bounding the price of stability and has been extended to other games as well; see [3,5]. For directed graphs, the bound of H_n was also proved to be tight [2]. Although the upper bound proof carries over to undirected network design games, the lower bound does not. The bound of H_n is the only known upper bound for multi-source network design games in undirected graphs. Better upper bounds are known for single-source games. For broadcast games, Fiat *et al.* [7] proved an upper bound of $O(\log \log n)$ while Li [9] presented an upper bound of $O(\log n/ \log \log n)$ for multicast games. These bounds are not known to be tight either and, actually, the gap with the corresponding lower bounds is large. For single-source games, in the full version of [7] Fiat *et al.* present a lower bound of $12/7 \approx 1.714$; their construction uses a broadcast game. This was the best lower bound known for the multi-source case as well until the recent work of Christodoulou *et al.* [6] who presented an improved lower bound of $42/23 \approx 1.826$. Higher (i.e., super-constant) lower bounds are only known for weighted variants of network design games (see [1,4]).

We present better lower bounds for general undirected network design games, as well as for the restricted variants of broadcast and multicast games. For the general case, we present a game that has price of stability at least $348/155 \approx 2.245$, improving the previously best known lower bound of [6]. Our proof uses a simple gadget as the main building block which is augmented by a recursive construction to our lower bound instance. The particular recursive construction of the game has two advantages. Essentially, the recursive construction blows up the price of stability of the gadget used as the main building block. Furthermore, recursion allows to handle successfully the technical difficulties in the analysis. We believe that our construction could be extended to use more complicated gadgets as building blocks that would probably lead to better lower bounds on the price of stability (at the expense of significantly more complicated proofs compared to our current one). For multicast games, we present a lower bound of 1.862. Our proof uses a game on a graph with a particular structure. For this game, we prove sufficient conditions on the edge costs of the graph so that a particular assignment is the unique Nash equilibrium. Then, the construction that yields the lower bound is the solution of a linear program which has the edge costs as variables, the sufficient conditions as constraints, an additional constraint that upper-bounds the optimal cost by 1, and its objective is to maximize the cost of the unique Nash equilibrium. The particular lower bound was obtained in a game with 100 players using the linear programming solver of Matlab. A slight variation of this construction yields our lower bound of 1.818 for broadcast games. In this case, we are able to obtain a more compact set of sufficient conditions so that there is a unique Nash equilibrium. As a result, we present a formal proof that the price of stability approaches $20/11 \approx 1.818$ when the number of players is large.

We remark that proving lower bounds on the price of stability is significantly more difficult for undirected network design games compared to their directed counterparts. This is due to the fact that edges are not constrained to be used in a single direction and, hence, the strategy space of each player is much larger. In order to prove high lower bounds, one must construct a game on an undirected graph in which any Nash equilibrium is much different than the optimal solution. Usually (this applies to all previous proofs as well as to the ones in the current paper), such lower bound constructions have a unique Nash equilibrium. Achieving simultaneously uniqueness (among many different possible assignments) and high cost of a Nash equilibrium is the most difficult part of our proofs.

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