On Linear Congestion Games with Altruistic Social Context

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Congestion games are, perhaps, the most famous class of non-cooperative games due to their capability to model several interesting competitive scenarios, while maintaining some nice properties. In these games there is a set of players sharing a set of *resources*, where each resource has an associated *latency function* which depends on the number of players using it (the so-called *congestion*). Each player has an available set of strategies, where each strategy is a non-empty subset of resources, and aims at choosing a strategy minimizing her cost which is defined as the sum of the latencies experienced on all the selected resources.

Congestion games have been introduced by Rosenthal [11]. He proved that each such a game admits a bounded *potential function* whose set of local minima coincides with the set of *pure Nash equilibria* of the game, that is, strategy profiles in which no player can decrease her cost by unilaterally changing her strategic choice. This existence result makes congestion games particularly appealing especially in all those applications in which pure Nash equilibria are elected as the ideal solution concept.

In these contexts, the study of the inefficiency of pure Nash equilibria, usually measured by the sum of the costs experienced by all players, has affirmed as a fervent research direction. To this aim, the notions of *price of anarchy* (Koutsoupias and Papadimitriou [9]) and *price of stability* (Anshelevich *et al.* [1]) are widely adopted. The price of anarchy (resp. stability) compares the performance of the worst (resp. best) pure Nash equilibrium with that of an optimal cooperative solution.

To the best of our knowledge, Chen and Kempe [5] were the first to study the effects of altruistic (and spiteful) behavior on the existence and inefficiency of pure Nash equilibria in some well-understood non-cooperative games. They focus on the class of non-atomic congestion games, where there are infinitely many players each contributing for a negligible amount of congestion, and show that price of anarchy decreases as the degree of altruism of the players increases.

Hoefer and Skopalik [7] consider (atomic) linear congestion games with γ_i-altruistic players, where $\gamma_i \in [0,1]$, for each player i. According to their model, player i aims at minimizing a function defined as $1-\gamma_i$ times her cost plus γ_i times the sum of the costs of all the players in the game (*also counting* player i). They show that pure Nash equilibria are always guaranteed to exist via a potential function argument, while, in all the other cases in which existence is not guaranteed, they study the complexity of the problem of deciding whether a pure Nash equilibrium exists in a given game.

Given the existential result by Hoefer and Skopalik [7], Caragiannis *et al.* [4] focus on the impact of altruism on the inefficiency of pure Nash equilibria in linear congestion games with altruistic players. However, they consider a more general model of altruistic behavior: in fact, for a parameter $\gamma_i \in [0,1]$, they model a γ_i-altruistic player i as a player who aims at minimizing a function defined as $1-\gamma_i$ times her cost plus γ_i times the sum of the costs of all the players in the game *other than* i[1]. In such a way, the more γ_i increases, the more γ_i-altruistic players tend to favor the interests of the others to their own ones, with 1-altruistic and 0-altruistic players being the two opposite extremal situations in which players behave in a completely altruistic or in a completely selfish way, respectively. Caragiannis *et al.* [4] consider the basic case of $\gamma_i = \gamma$ for each player i and show that the price of anarchy is $\frac{5-\gamma}{2-\gamma}$ for $\gamma \in [0,1/2]$ and $\frac{2-\gamma}{1-\gamma}$ for $\gamma \in [1/2,1]$ and that these bounds hold also for load balancing games. This result appears quite surprising, because it shows that altruism can only have a harmful effect on the efficiency of linear congestion games, since the price of anarchy increases from 5/2 up to an unbounded value as the degree of altruism goes from 0 to 1. On the positive side, they prove that, for the special case of symmetric load balancing games, the price of anarchy is $\frac{4(1-\gamma)}{3-2\gamma}$ for $\gamma \in [0,1/2]$ and $\frac{3-2\gamma}{4(1-\gamma)}$ for $\gamma \in [1/2,1]$, which shows that altruism has a beneficial effect as long as $\gamma \in [0,0.7]$. Note that, that for $\gamma = 1/2$, that is when selfishness and altruism are perfectly balanced, the price of anarchy drops to 1 (i.e., all pure Nash equilibria correspond to socially optimal solutions), while, as soon as γ approaches

[1]Note that each game with γ_i-altruistic players, where $\gamma_i \in [0,1]$, in the model of Hoefer and Skopalik [7] maps to a game with γ_i'-altruistic players, where $\gamma_i' \in [0,1/2]$, in the model of Caragiannis *et al.* [4].

1, the price of anarchy again grows up to an unbounded value.

Recently, de Keijzer *et al.* [6] proposed a model for altruistic and spiteful behavior further generalizing the one of Caragiannis *et al.* [4]. According to their definition, each non-cooperative game with n players is coupled with a real matrix $\Gamma = (\gamma_{ij}) \in \mathbb{R}^{n \times n}$, where γ_{ij} expresses how much player i cares about player j. In such a framework, player i wants to minimize the sum, for each player j in the game (thus also counting i), of the cost of player j multiplied by γ_{ij}. Thus, a positive (resp. negative) value γ_{ij} expresses an altruistic (resp. spiteful) attitude of player i towards player j[2]. When considering linear congestion games with altruistic players, along the lines of the negative results of Caragiannis *et al.* [4], as soon as there are two players i, j such that $\gamma_{ij} > \gamma_{ii}$, i.e., player i cares more about player j than about herself, the price of anarchy becomes unbounded. Therefore, Keijzer *et al.* [6] focus on the scenario, which they call *restricted altruistic social context*, in which $\gamma_{ii} \geq \gamma_{ij}$ for each pair of players i and j. By extending the smoothness framework of Roughgarden [12], they show an upper bound of 7 on the price of anarchy of coarse correlated equilibria, which implies the same upper bound also on the price of anarchy of correlated equilibria, mixed Nash equilibria and pure Nash equilibria (whenever the latter exist). Moreover, they prove that, when restricting to load balancing games with identical resources, such an upper bound decreases to $2+\sqrt{5} \approx 4.236$. Very recently, Rahn and Scäfer [10] improved the 7 upper bound to 17/3 and showed that this is tight even for pure Nash equilibria.

Noting that matrix Γ implicitly represents the *social context* (for instance, a social network) in which the players operate, the model of de Keijzer *et al.* [6] falls within the scope of the so-called *social context games*. In these games, the payoff of each player is redefined as a function, called *aggregating function*, of her cost and of those of her neighbors in a given *social context graph*.

Social context games have been introduced and studied by Ashlagi, Krysta, and Tennenholtz [2] for the class of load balancing games, in the case in which the aggregating function is one among the minimum, maximum, sum and ranking functions, for which they gave an almost complete characterization of the cases in which existence of pure Nash equilibria is guaranteed. The model of de Keijzer *et al.* [6], hence, coincides with a social context game in which the social context graph has weighted edges and the aggregating function is a weighted sum. The issues of existence and inefficiency of pure Nash equilibria for the case of social context linear congestion games have been considered by Bilò *et al.* [3]. In particular, for the aggregating function sum, pure Nash equilibria are shown to exist for each social context graph via an exact potential function argument and the price of anarchy is shown to fall within the interval $[5; 17/3]$.

We consider the issues of existence and inefficiency of pure Nash equilibria in linear congestion games with social context as defined by de Keijzer *et al.* [6]. In particular, we restrict our attention to the case of altruistic players, that is, the case in which the matrix Γ has only non-negative entries. Hoefer and Skopalik [8] had shown that pure Nash equilibria are always guaranteed to exist via an exact potential function argument when either the altruistic social context is restricted and Γ is symmetric. We prove that both properties are essential to guarantee an existential result, by providing instances with three players not admitting pure Nash equilibria as soon as exactly one of them is not satisfied.

We then show that, in the restricted altruistic social context, the price of anarchy of coarse correlated equilibria remains 17/3 even in the special case of load balancing games. Such a result, which improves the one given by Rahn and Schäfer [10], proves that the assumption of having identical resources is essential in the upper bound of $2 + \sqrt{5}$ given by de Keijzer *et al.* [6] for the case of load balancing games.

For the price of stability in the restricted altruistic social context, we give an upper bound of 2 holding for each symmetric matrix Γ and a lower bound of $1 + 1/\sqrt{2} \approx 1.707$ holding for the case in which Γ is a boolean symmetric matrix.

Finally, we also consider the special case in which Γ is such that $\gamma_{ij} = \gamma_i$ for each pair of indexes i, j with $i \neq j$, which coincides with the general model of γ_i-altruistic players of Caragiannis *et al.* [4]. We show that pure Nash equilibria are always guaranteed to exist in any case via an exact potential function argument (this slightly improves the existential result by Hoefer and Skopalik [7] since they only proved the existence of a weighted potential function) and give an upper bound on the price of anarchy in the general case and an exact bound on the price of stability when $\gamma_i = \gamma$ for each player i.

[2]Nevertheless, a model in which $\gamma_{ii} = 1$ and $\gamma_{ij} = \gamma_{ji}$ for each $i, j \in [n]$ had been previously considered by Hoefer and Skopalik in [8].

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