Weighted Calderón-Zygmund and Rellich inequalities in L^p

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In 1956, Rellich proved the inequalities

$$\left(\frac{N(N-4)}{4}\right)^2 \int_{\mathbb{R}^N} |x|^{-4}|u|^2\,dx \le \int_{\mathbb{R}^N} |\Delta u|^2\,dx$$

for $N \neq 2$ and for every $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$, see [10]. These inequalities have been then extended to L^p-norms: in 1996, Okazawa proved the validity of

$$\begin{aligned}&\left(\frac{N}{p}-2\right)^p \left(\frac{N}{p'}\right)^p \int_{\mathbb{R}^N} |x|^{-2p}|u|^p\,dx \\ \le\ &\int_{\mathbb{R}^N} |\Delta u|^p\,dx\end{aligned}$$

for $1 < p < \frac{N}{2}$ (see [9] and also [5]) showing also the optimality of the constants.

Weighted Rellich inequalities have also been studied. In 1998, Davies and Hinz ([2, Theorem 12]) obtained for $N \ge 3$ and for $2-\frac{N}{p} < \alpha < 2-\frac{2}{p}$

$$\begin{aligned}&C(N,p,\alpha)\int_{\mathbb{R}^N} |x|^{(\alpha-2)p}|u|^p\,dx \\ \le\ &\int_{\mathbb{R}^N} |x|^{\alpha p}|\Delta u|^p\,dx\end{aligned} \tag{1}$$

with the optimal constants

$$C(N,p,\alpha) = \left(\frac{N}{p}-2+\alpha\right)^p \left(\frac{N}{p'}-\alpha\right)^p.$$

Later Mitidieri showed that (1) holds in the wider range $2-\frac{N}{p} < \alpha < N-\frac{N}{p}$ and with the same constants, see [7, Theorem 3.1].
In recent papers Ghoussoub and Moradifam and Caldiroli and Musina, see [4], [1], improved weighted Rellich inequalities for $p = 2$ by giving necessary and sufficient conditions on α for the validity of (1) and finding also the optimal constants $C(N,2,\alpha)$. In particular in [1] it is proved that (1) is verified for $p = 2$ if and only if $\alpha \neq N/2+n$, $\alpha \neq -N/2+2-n$ for every $n \in \mathbb{N}_0$. This approch makes use of the so called Emden-Fowler transform which reduces the operator $|x|^\alpha\Delta$ in $\mathbb{R}^N$ to a uniforly elliptic operator in the cylinder $\mathbb{R}\times S^{N-1}$ and Rellich inequalities to spectral inequalities for the Laplace Beltrami Δ_0 on S^{N-1}. We also refer to [4, Section 3] where results similar to [1] have been obtained under the restriction $\alpha \ge (4-N)/2$ and with different methods.

In this paper we extend the results in [1], [4] to $1 \le p \le \infty$, computing also best constants in some cases. We show that (1) holds if and only if $\alpha \neq N/p'+n$, $\alpha \neq -N/p+2-n$ for every $n \in \mathbb{N}_0$. Moreover, we use Rellich inequalities to find necessary and sufficient conditions for the validity of weighted Calderón-Zygmund estimates when $1 < p < \infty$

$$\int_{\mathbb{R}^N} |x|^{\alpha p}|D^2u|^p\,dx \le C\int_{\mathbb{R}^N} |x|^{\alpha p}|\Delta u|^p\,dx \tag{2}$$

for $u \in C_c^\infty(\mathbb{R}^N\setminus\{0\})$.
Weighted Calderón-Zygmund inequalities are well-known in the literature, in the framework of singular integrals. In 1957 Stein (see [11]) proved the inequalities

$$\||x|^\alpha Tf\|_p \le C\||x|^\alpha f\|_p \tag{3}$$

for $1 < p < \infty$, $-N/p < \alpha < N/p'$, where T is the Calderón-Zygmund kernel corresponding to the operator $D^2\Delta^{-1}$. Subsequent generalizations of the above result can be found in the papers of Kree, Muckenhoupt and Wheeden (see [3], [8]) where more general kernels are treated. Taking $u \in C_c^\infty(\mathbb{R}^N\setminus\{0\})$ and setting $f = \Delta u$, inequalities (3) imply that

$$\||x|^\alpha D^2u\|_p \le C\||x|^\alpha \Delta u\|_p.$$

However the last inequalities can hold also when (3) fail, that is outside of the range $-N/p < \alpha < N/p'$, since f has compact support whenever u has but the converse is clearly false. In particular, the condition $\alpha > -N/p$ is needed for the integrability of $|x|^\alpha Tf$ near the origin, whereas $\alpha < N/p'$ is needed for the integrability at infinity, if Tf behaves like $|x|^{-N}$. We find that (2) holds if and only if $\alpha \neq N/p'+n$ for every $n \in \mathbb{N}_0$ and , $\alpha \neq -N/p+2-n$ for every $n \in \mathbb{N}, n \ge 2$.
We consider also more general operators

$$L = \Delta + c\frac{x}{|x|^2}\cdot\nabla - \frac{b}{|x|^2}$$

with $b, c \in \mathbb{C}$ and investigate the validity of weighted Rellich inequalities of the form

$$\begin{aligned} & C(N,p,\alpha,b,c) \int_{\mathbb{R}^N} |x|^{(\alpha-2)p} |u|^p \, dx \\ \leq & \int_{\mathbb{R}^N} |x|^{\alpha p} |Lu|^p \, dx \end{aligned} \tag{4}$$

for $u \in C_c^\infty(\mathbb{R}^N \setminus \{0\})$ and $1 \leq p < \infty$. We prove necessary and sufficient conditions on α for the validity of (4) and, in certain cases, we explicitely compute the best constants.

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