Montel resolvents and uniformly mean ergodic semigroups of linear operators

A.A. Albanese, [a] José Bonet,[b] Werner J. Ricker[c]

[a]Dipartimento di Matematica e Fisica “E. De Giorgi”, Università del Salento, Italy

[b]Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de Valencia, Spain

[c]Math.-Geogr. Fakultät, Katholische Universität Eichstätt-Ingolstadt, Germany

Let $(T(t))_{t\geq 0}$ be a C_0-semigroup of continuous linear operators in a locally convex Hausdorff space X (briefly, lcHs). Ergodic theorems have a long tradition and are usually formulated for the Cesàro averages $C(r)x = \frac{1}{r}\int_0^r T(t)x\,dt$ or the Abel averages $\lambda R(\lambda, A)x = \lambda\int_0^\infty e^{-\lambda t}T(t)x\,dt$, for $x \in X$, where $r \to \infty$ and $\lambda \to 0+$, respectively. In the former case one speaks of the mean ergodicity of $(T(t))_{t\geq 0}$ and in the latter case of its Abel mean ergodicity. Particularly well developed is the theory and its applications when X is a Banach space, both for the strong operator topology τ_s-convergence of $\lim_{r\to\infty} C(r)$, resp. $\lim_{\lambda\to 0+}\lambda R(\lambda, A)$, and for their operator norm convergence. For certain aspects of the theory of mean ergodic semigroups of operators in non-normable spaces X (mainly for τ_s) we refer to [3], [5, Chapter 2], [6, Chapter III, Section 7] and the references therein. Further results, involving $\{C(r)\}_{r\geq 0}$, occur in [2], where geometric features of the underlying space X also play an important role. But for a few exceptions, there are not so many results available concerning the mean ergodicity of C_0-semigroups of operators in lcHs' when the averages are required to converge for the topology τ_b of uniform convergence on the bounded subsets of X. The aim of this paper is to develop this topic further.

Many criteria concerning the mean ergodicity of a τ_s-continuous C_0-semigroup $(T(t))_{t\geq 0}$ acting in a lcHs X involve its infinitesimal generator A. Under mild conditions this is a closed operator with a dense domain $D(A) \subseteq X$. For X a Banach space the resolvent set $\rho(A)$ of A is an open, non-empty subset of $\mathbb{C}$ and so the well developed spectral theory of closed operators in such spaces is available. In particular, the resolvent map $\lambda \to R(\lambda; A) := (\lambda - A)^{-1}$ of A is holomorphic in $\rho(A)$ for the operator norm topology. For X non-normable, the spectral theory of closed operators A is much less developed. Even if A is the infinitesimal generator of a τ_s-continuous C_0-semigroup in a Fréchet space X and $D(A) = X$, it can happen that $\rho(A)$ fails to be open in $\mathbb{C}$ in which case the question of $R(\cdot; A)$ being holomorphic is not well-posed. In Section 3 of this paper we investigate and develop those aspects of spectral theory for closed operators (in spaces which may be non-normable) and, in particular, for infinitesimal generators, which are needed in later sections. In Banach spaces there is a close connection between operator norm continuous, mean ergodic C_0-semigroups $(T(t))_{t\geq 0}$ and compactness of the resolvent operators $R(\lambda; A)$ of the infinitesimal generator A of $(T(t))_{t\geq 0}$, [4, Chapter V, Section 4]. For X a more general lcHs an appropriate analogue of $R(\lambda; A)$ being compact is that it maps bounded subsets of X to relatively compact subsets of X; such operators are called Montel. In Section 4 of this paper we investigate the connections between the operators $R(\lambda; A)$ being Montel (assuming $\rho(A) \neq \emptyset$), the τ_b-continuity of the map $t \to T(t)$ in $[0; \infty)$ and of the individual operators $T(t)$, for $t \geq 0$, being Montel; this is made precise in the main result. In the last sections of the paper we treat some continuity and spectral properties of general C_0-semigroups of operators and their infinitesimal generators. These results are needed to turn our attention to mean ergodic features of C_0-semigroups $(T(t))_{t\geq 0} \subset \mathcal{L}(X)$, with X a sequentially complete lcHs and $\mathcal{L}(X)$ the vector space of all continuous linear operators from X into itself. Under mild conditions, the Cesàro averages $\{C(r)\}_{r\geq 0} \subset \mathcal{L}(X)$ exist as do the Abel averages $\{\lambda R(\lambda; A) \mid \lambda > 0\} \subset \mathcal{L}(X)$ where, for each real $\lambda > 0$, the resolvent operator $R(\lambda; A)$ coincides with the operator $R_\lambda x =: \int_0^\infty e^{-\lambda t}T(t)x\,dt$, for $x \in X$, mentioned above and which is defined via X-valued Riemann integrals. The central notions are the uniform mean ergodicity (resp. uniform Abel mean ergodicity) of $(T(t))_{t\geq 0}$, that is, τ_b-$\lim_{r\to\infty} C(r)$ (resp. τ_b-$\lim_{\lambda\to 0+}\lambda R(\lambda; A)$) exists. As already mentioned, in Banach spaces many results are available which imply or are equivalent to $(T(t))_{t\geq 0}$ being uniformly mean ergodic. But, for non-normable X, not so much is known. In the last section of the paper we present several new results in this direction.

REFERENCES

1. A.A. Albanese, J. Bonet, W.J. Ricker, *Montel resolvents and uniformly mean ergodic semigroups of linear operators*, Quaest. Math. 36 (2013), 253-290.
2. A.A. Albanese, J. Bonet, W.J. Ricker, *Mean ergodic semigroups of operators*, Rev. R. Acad. Cien. Serie A, Mat., RACSAM 106 (2012), 299-319.
3. W.F. Eberlein, *Abstract ergodic theorems and weak almost periodic functions*, Trans. Amer. Math. Soc. 67 (1949), 217-240.
4. K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Springer, New-York, 1999.
5. U. Krengel, *Ergodic Theorems*, Walter de Gruyter, Berlin, 1985.
6. H.H. Schaefer, *Banach Lattices and Positive Operators*, Springer, Berlin/ Heidelberg, 1974.