Convergence of arithmetic means of operators in Fréchet spaces

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The purpose of this paper is to investigate the behaviour of the sequence of arithmetic means $T_{[n]} := \frac{1}{n}\sum_{m=1}^{n} T^m$ of the iterates $T^m := T \circ \ldots \circ T$ of a continuous linear operator $T \in \mathcal{L}(X)$ on a Fréchet space X. A useful result of Lin [6] asserts that the following conditions are equivalent for an operator T (on a Banach space X) which satisfies $\lim_{n\to\infty} \|T_n/n\| = 0$.
(1) T is uniformly mean ergodic, i.e., there is $P \in \mathcal{L}(X)$ with $\lim_{n\to\infty} \|T_{[n]} - P\| = 0$.
(2) The range $(I-T)(X)$ is closed and $X = \mathrm{Ker}(I-T) \oplus (I-T)(X)$.
(3) $(I-T)^2(X)$ is closed.
(4) $(I-T)(X)$ is closed.

It was observed in Example 2.17 of [2] that there exist power bounded, uniformly mean ergodic operators T on the Fréchet space s of rapidly decreasing sequences for which $(I-T)(X)$ is not closed. On the other hand, Theorem 4.1 of [3] provides an extension of Lin's result to those Fréchet spaces X which are quotients of countable products of Banach spaces (the so called quojections), under the additional assumption that $\mathrm{Ker}(I-T) = \{0\}$. In the present paper we undertake a careful analysis of the possible extension of Lin's result to the setting of Fréchet spaces. First, we show that every Montel Köthe echelon space $\lambda_p(A)$ of order $p \in [1,\infty) \cup \{0\}$ not isomorphic to a countable product of copies of the scalar field admits an operator $T \in \mathcal{L}(\lambda_p(A))$ which is power bounded and uniformly mean ergodic, but such that $I-T$ is not surjective and has dense range. The same result also holds if $\lambda_p(A)$ is non-normable, admits a continuous norm and satisfies the density condition. In contrast to these results, we prove that the conditions (1)–(4) above are equivalent for operators T defined on a Fréchet space X which does not have a separated quotient which is isomorphic to a nuclear Köthe echelon space with a continuous norm. These spaces, called prequojections, are precisely those Fréchet spaces whose strong bidual is a quojection. As a concrete example we investigate the mean ergodic properties of the classical Cesàro operator

$$C(x) = \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)_n, \quad x = (x_n)_n \in \mathbb{C}^{\mathbb{N}},$$

in the quojection Fréchet space $\mathbb{C}^{\mathbb{N}}$ of all sequences, as well as in the Banach sequence spaces c_0, c, ℓ^p $(1 < p \leq \infty)$, bv_0 and bv_p $(1 \leq p < \infty)$. Finally, in the last section of this paper, inspired by results in [4,5], we investigate when the identity

$$\left\{x \in X \mid \{\sum_{k=1}^{n} T^k x\}_n \text{ is a bounded sequence in } X\right\} = (I-T)(X),$$

called Browders equality, holds for a power bounded operator $T \in \mathcal{L}(X)$ in a locally convex space X. The main results of this section establish the connection of Browders equality to uniform mean ergodicity.

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