Convergence of arithmetic means of operators in Fréchet spaces

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The purpose of this paper is to investigate the behaviour of the sequence of arithmetic means $T_{[n]} := \frac{1}{n} \sum_{m=1}^{n} T^{m}$ of the iterates $T^{m} := T \circ \ldots \circ T$ of a continuous linear operator $T \in \mathcal{L}(X)$ on a Fréchet space $X$. A useful result of Lin [6] asserts that the following conditions are equivalent for an operator $T$ (on a Banach space $X$) which satisfies $\lim_{n \to \infty} \|T_{n}/n\| = 0$.

1. $T$ is uniformly mean ergodic, i.e., there is $P \in \mathcal{L}(X)$ with $\lim_{n \to \infty} \|T_{n} - P\| = 0$.
2. The range $(I - T)(X)$ is closed and $X = \text{Ker}(I - T) \oplus (I - T)(X)$.
3. $(I - T)^{2}(X)$ is closed.
4. $(I - T)(X)$ is closed.

It was observed in Example 2.17 of [2] that there exist power bounded, uniformly mean ergodic operators $T$ on the Fréchet space $s$ of rapidly decreasing sequences for which $(I - T)(X)$ is not closed. On the other hand, Theorem 4.1 of [3] provides an extension of Lin’s result to those Fréchet spaces $X$ which are quotients of countable products of Banach spaces (the so called quojections), under the additional assumption that $\text{Ker}(I - T) = \{0\}$. In the present paper we undertake a careful analysis of the possible extension of Lin’s result to the setting of Fréchet spaces. First, we show that every Montel Köthe echelon space $\lambda_{p}(A)$ of order $p \in [1, \infty) \cup \{0\}$ not isomorphic to a countable product of copies of the scalar field admits an operator $T \in \mathcal{L}(\lambda_{p}(A))$ which is power bounded and uniformly mean ergodic, but such that $I - T$ is not surjective and has dense range. The same result also holds if $\lambda_{p}(A)$ is non-normable, admits a continuous norm and satisfies the density condition. In contrast to these results, we prove that the conditions (1)–(4) above are equivalent for operators $T$ defined on a Fréchet space $X$ which does not have a separated quotient which is isomorphic to a nuclear Köthe echelon space with a continuous norm. These spaces, called prequojections, are precisely those Fréchet spaces whose strong bidual is a quojection.

As a concrete example we investigate the mean ergodic properties of the classical Cesàro operator

$$C(x) = \left( \frac{1}{n} \sum_{k=1}^{n} x_{k} \right)_{n}, \quad x = (x_{n})_{n} \in \mathbb{C}^{N},$$

in the quojection Fréchet space $\mathbb{C}^{N}$ of all sequences, as well as in the Banach sequence spaces $c_{0}$, $c$, $\ell^{p}$ ($1 < p \leq \infty$), $b_{0}$ and $b_{p}$ ($1 \leq p < \infty$). Finally, in the last section of this paper, inspired by results in [4,5], we investigate when the identity

$$\{ x \in X \mid \{ \sum_{k=1}^{n} T^{k}x \} \text{ is a bounded sequence in } X \} = (I - T)(X),$$

called Browder’s equality, holds for a power bounded operator $T \in \mathcal{L}(X)$ in a locally convex space $X$. The main results of this section establish the connection of Browders equality to uniform mean ergodicity.

REFERENCES
