# Geometric aspects of symmetries and conservation laws in relativistic mechanics

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Relativistic mechanics of particles with mass is usually formulated, in a differential geometric context, as a theory on the tangent bundle of spacetime endowed with the symplectic form induced by the pseudo-Riemannian metric. Anyway, this formulation has the drawback of presenting a degenerate Lagrangian. The degeneracy is due to the fact that the Lagrangian is invariant with respect to the affine reparametrization of curves, which act as a gauge group of the theory [ 1]. This problem can be solved through the introduction of suitable Dirac constraints *e.g.* mass shells.

Recently, a differential-geometric formulation of relativistic mechanics based on jets of submanifolds and contact forms has been presented [ 2, 3]. Such a geometric model provides a universal mass shell constraint as a generalized contact manifold, according with [ 4]. Since contact manifold are an intensively studied topic in differential geometry, it seems interesting to apply techniques from this area in general relativistic mechanics. In particular, a characterization and classification of symmetries of generalized contact manifolds turns produces interesting mathematical results concerning relativistic mechanics.

During 2012 J. Janyška and R. Vitolo completed a symmetry analysis of the new geometric objects introduced in this model [ 5], in the same spirit as [ 6, 7, 8]. We were able to prove that all (infinitesimal) symmetries of the generalized contact structure which project to a spacetime symmetry boil down to infinitesimal isometries which are also symmetries of the electromagnetic field of the theory. Moreover, we were able to identify symmetries of the generalized contact structure with a certain class of conserved quantities of the equation of particle motions. The Lie algebra of symmetries turn out to be isomorphic to the Lie algebra of *special phase functions* endowed with the (degenerate) Poisson bracket. Such functions have the coordinate expression

$$f = -\frac{c}{\sqrt{|g_{00} + 2g_{0i}x^i_0 + g_{ij}x^i_0 x^j_0|}}(g_{0\rho}+x^i_0 g_{i\rho})X^\rho + f_o,$$

where $X$ is a symmetry of the metric and the electromagnetic field and $i_X F = -df_0$.

We were also able to define a momentum map for such a structure. Momentum maps are used for symplectic/contact reduction of mechanical systems, and it could be an interesting direction of research to study the reduction of relativistic mechanics and its interplay with quantization.

Another direction in which we are doing research is that of *hidden symmetries*. There are conserved quantities which depend in polynomial way on velocities, so that these quantities do not correspond to a symmetry of spacetime. Indeed, the corresponding infinitesimal transformations do not project onto a spacetime symmetry. We were able to prove that such symmetries coincide with symmetries of the generalized contact structure that do not project to spacetime symmetries.

The field of hidden symmetries is quite active (see, *e.g.* [ 9]). Our wish is to find general classification results of hidden symmetries, as we did for projectable symmetries, as well as applications to particular spacetimes, possibly through symbolic computations.

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