On the geometry of Hamiltonian formalism for partial differential equations

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The integrability of Partial Differential Equations (PDEs) is quite often obtained from the existence of distinguished algebraic and/or geometric structures. More particularly, a feature of many (non-linear) integrable differential equations is the existence of an infinite sequence ('hierarchy') of commuting higher (or generalized) symmetries and/or conservation laws. Such sequences are often generated through differential operators. The algebraic-geometric analysis of PDEs in 2 independent variables and 1 dependent variable produced many results. In this context, some new results on the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equation

$$f_{ttt} = f_{xxt}^2 - f_{xxx}f_{xtt} \qquad (1)$$

were obtained in [1], in the framework of a new geometric approach to integrability differential operators developed in [2].

Currently, interesting problems related to integrability differential operators come from multidimensional (*i.e.*, with more than 2 independent variables) and multicomponent (*i.e.*, with more than 1 dependent variable) systems of PDEs.

In the multidimensional area, a joint research by J.S. Krasil'shchik, A.M. Verbovetsky (Independent University of Moscow and R. Vitolo [3, 4] shown that integrability differential operators are rare and very difficult to compute. The difficulty is intrinsically related in exponential way to the number of variables involved in computations. The computations which are necessary in order to solve equations for recursion or Hamiltonian operators can be managed only through symbolic software. For this reason P.H.M. Kersten and the symbolic software group of the University of Twente, and later R. Vitolo, developed the program CDIFF that runs in the symbolic environment REDUCE (which is now free software). This is one of the few publicly available programs which is able to make computations on integrability differential operators. Computations for multidimensional PDEs proved to be so hard to push symbolic software to its limits; this led, in cooperation with REDUCE developers to a general improvement of the program itself. R. Vitolo now is one of REDUCE developers, see `http://sourceforge.net/projects/reduce-algebra/`. More details on CDIFF, a user guide and many examples of use can be found here [5].

In the multicomponent area, in the case of two independent variables, a cooperation of M. Pavlov (Lebedev Institute of Theoretical Physics) and R. Vitolo considered the WDVV equation in 3 components.

$$u^1_t = \frac{1}{2}(u^2u^3 - u^1u^2 - u^1u^3)_x,$$
$$u^2_t = \frac{1}{2}(u^1u^3 - u^2u^1 - u^2u^3)_x,$$
$$u^3_t = \frac{1}{2}(u^1u^2 - u^3u^1 - u^3u^2)_x.$$

It was shown that the above equation, besides the two Hamiltonian structures presented in [6], gives rise through a nonlocal change of variables to a hierarchy of equations in which at least the first two members have a Lagrangian representation, according with [7]. Moreover, it was found an interesting relation between: conservation law densities quadratic in x-velocities and x-accelerations; coefficients of the above Lagrangian representation; coefficients of the second Hamiltonian operator. These results have been communicated [8], but are still unpublished.

The methods used here allows us to generalize the construction to PDEs which have similar geometric structures. More precisely, starting from a multicomponent system of PDEs which has a single first order Hamiltonian operator with constant coefficients it should be possible to look for a second Hamiltonian operator provided the PDE has enough conservation laws. A 6-component WDVV equation, for which only one constant-coefficient first-order Hamiltonian structure is known, is under study, and conservation law densities quadratic in x-velocities and x-accelerations have already been found.

The problem of defining the class of equations to which the method is applicable and finding new examples is currently under investigation.

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