Lie identities for skew and symmetric elements of semiprime superalgebras with superinvolution

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Let $A = A_0 \oplus A_1$ be an associative superalgebra over a commutative unital ring of scalars R such that $\frac{1}{2} \in R$. An element a of A is said to be *homogeneus* (of degree i) if $a \in A_i$ (and we write $\overline{a} = i$). Let us denote by A^- the Lie superalgebra obtained from A via the Lie superbracket $[a,b]_s := ab - (-1)^{\overline{a}\overline{b}} ba$, for all homogeneus elements $a, b \in A$ (the expression extends over the rest of the elements by linearity). If A has a superinvolution $*$, let K be the subalgebra of the Lie superalgebra A^- consisting of *skew elements* of A with respect to $*$, namely $K := \{a \,|\, a \in A,\ a^* = -a\}$. When A is *trivial*, i.e. $A_1 = 0$, A is nothing but an associative algebra with involution and the Lie superbracket $[\,,\,]_s$ coincides with the usual Lie bracket $[\,,\,]$. In this setting, an interesting question is to decide if crucial information on the algebraic structure of A can be deduced from properties of A^- or K. This interplay has been the subject of a good deal of attention over the decades.

In the last years the relation between A, A^- and K has been profusely investigated by several authors for *non-trivial* superalgebras as well. The motivations for this line of research mainly come from the classification of the finite-dimensional simple Lie superalgebras over an algebrically closed field of characteristic zero given by Kac [6]. In fact, we can find in it examples that are superalgebras of skew elements with respect to a superinvolution in a simple associative superalgebra or Lie superalgebras associated to a simple associative superalgebra. This result suggests that the structure of A as associative superalgebra and A^- and K as Lie superalgebras could be related. In this direction, Gómez-Ambrosi and Shestakov [5] studied the Lie structure of K when A is simple. Later these results were extended to the context of prime and semiprime associative superalgebras in [3] and [**?**], respectively. More recently, Laliena and Sacristán explored the structure of semiprime associative superalgebras with superinvolution under certain additional regularity condition on symmetric and skew elements ([9]) and when $[K^2, K^2]_s = 0$ ([8]). We notice that the Lie structure of simple and prime associative superalgebras without superinvolution was previously studied by Montgomery ([12]) and Montaner ([11]), respectively.

On the other hand, one can consider the Jordan superalgebra A^+ obtained from A via the circle operation $a \circ_s b := ab + (-1)^{\overline{a}\overline{b}} ba$ for all homogeneus elements a, b of A (also in this case, the expression extends over the rest of the elements by linearity). When A is equipped with a superinvolution $*$, let $H := \{a \,|\, a \in A,\ a^* = a\}$ be the subalgebra of the Jordan superalgebra A^+ consisting of *symmetric elements* of A with respect to $*$. In the Kac’s classification ([7]) of finite-dimensional simple Jordan superalgebras over an algebrically closed field of characteristic zero we find examples of simple Jordan superalgebras of the form A^+ and of the form $H(A,*)$, where A is a simple associative superalgebra (with a superinvolution $*$ in the latter case). This is one of the reasons for which A^+ and H have been the subject of a good deal of attention as well (we refer, for instance, to [2] and [4]).

The goal of this paper is to investigate semiprime associative superalgebras with superinvolution whose subspaces of skew elements or symmetric elements are *Lie nilpotent* or *Lie solvable*. We recall that a graded subspace S of a superalgebra A is said to be Lie nilpotent if, set $[x_1, \ldots, x_n]_s := [[x_1, \ldots x_{n-1}]_s, x_n]_s$ for all $n \geq 2$, there exists an integer m such that

$$[x_1, \ldots, x_m]_s = 0$$

for all $x_1, \ldots, x_m \in S$, and *Lie solvable* if, set $[x_1, x_2]_s^\circ := [x_1, x_2]_s$ and inductively

$$[x_1, \ldots, x_{2^{n+1}}]_s^\circ := [[x_1, \ldots, x_{2^n}]_s^\circ, [x_{2^n+1}, \ldots, x_{2^{n+1}}]_s^\circ]_s,$$

there exists an integer m such that

$$[x_1, \ldots, x_{2^{m+1}}]_s^\circ = 0$$

for all $x_1, \dots, x_{2^{m+1}} \in S$.

We notice that in every semiprime associative superalgebra A the intersection of all the prime ideals of A is zero. Consequently A is a subdirect product of its prime images. If each prime image of A is a central order in a simple superalgebra at most n^2-dimensional over its centre, we say that A is $S(n)$. This definition is required to state the main result on Lie solvability condition.

Theorem 1. *Let A be a non-trivial semiprime associative superalgebra over a commutative unital ring of scalars R such that $\frac{1}{2} \in R$ endowed with a superinvolution. If H is Lie solvable, then A is $S(2)$.*

We stress that if K is Lie solvable, so is H. Thus the result still holds by replacing H with K. Furthermore it is true if H is Lie nilpotent as well: indeed, the latter fact implies that H is Lie solvable (obviously, the same holds also for K, which has the structure of Lie superalgebra).

In the case in which H or K are Lie nilpotent, we are able to provide a characterization in terms of identities satisfied by the symmetric or skew elements of A.

Theorem 2. *Let A be a non-trivial semiprime associative superalgebra over a commutative unital ring of scalars R such that $\frac{1}{2} \in R$ endowed with a superinvolution. Then*

(a) H is Lie nilpotent if, and only if, the elements of H commute;

(b) K is Lie nilpotent if, and only if, the elements of K commute.

Classically, this situation has been studied in the context of semiprime algebras with involution by Giambruno and Sehgal (Theorem 1 of [1]) and Lee, Sehgal and Spinelli (Proposition 2.4 and 2.6 of [10], although there the authors consider algebras over fields, the statements still hold for algebras over rings). Their results can be summarized in the following

Theorem 3. *Let A be a semiprime associative algebra over a commutative unital ring of scalars R such that $\frac{1}{2} \in R$ endowed with an involution. The following statements are equivalent:*

(i) K is Lie nilpotent;

(ii) K is Lie solvable;

(iii) K is commutative.

Theorem 4. *Let A be a semiprime associative algebra over a commutative unital ring of scalars R such that $\frac{1}{2} \in R$ endowed with an involution. Then*

(a) H is Lie nilpotent if, and only if, H is commutative;

(b) H is Lie solvable if, and only if, H is Lie metabelian.

In particular, if H is Lie solvable, then A is $S(2)$.

We notice that only a partial superanalogous of Theorem 3 is obtained. In fact, in non-trivial superalgebras setting the Lie solvability of K does not imply the Lie nilpotency of K, not even if the superalgebra is simple. An easy example is provided by the superalgebra of (2×2)-matrices $M_{1,1}(F)$ over a field F of characteristic not 2 equipped with the transposition superinvolution. Furthermore, we cannot expect that the Lie nilpotency of K or H forces them to be supercommutative (namely, $[a, b]_s = 0$ for all a, b in K or H).

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