Homogeneous Calderon-Zygmund estimates for second order elliptic operators

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Given an uniformly elliptic operator

$$L = \sum_{i,j=1}^{N} a_{ij}(x) D_{ij},$$

with (a_{ij}) bounded and uniformly continuous in $\mathbb{R}^N$, a-priori estimates and solvability results in Sobolev spaces are well known in literature. The classical a-priori estimates

$$\|D^2 u\|_p \le C(\|Lu\|_p + \|u\|_p),$$

$u \in W^{2,p}(\mathbb{R}^N)$, solvability results for the equation $Lu - \lambda u = f$, $\lambda > 0$, can be found in [4].
In our knowledge, the validity of the homogeneous estimate

$$\|D^2 u\|_p \le C\|Lu\|_p \tag{1}$$

has been less exploited. We investigated about the validity of the homogeneous estimates above mentioned. We proved that, if the second order coefficients admit limit at infinity, then (1) is verified.
We gave also an example where the above homogeneous estimates fail.
We also proved that, under the same assumption on the coefficients, the equation $Lu = f$ is uniquely solvable in L^p spaces. We mention that similar results concerning the homogenous estimates have been already proved in certain special cases. Krylov proved (1) when the coefficients depend only on one coordinate, or even when the coefficients depend on two coordinates but p is close to 2. Avellaned and Lin showed the homogeneous estimates for second order operators in divergence form with periodic coefficients ([1]). Related results can also be found in [7].

In order to prove our result, we introduced the one-parameter family of operators L_t given by

$$L_t = (1-t)L_0 + tL$$

where L_0 is the constant coefficients operator having the limit values at infinity of $(a_{ij})_{i,j=1}^N$ as coefficients. The first step consisted in proving, by using the classical homogeneous a-priori estimates for L_0 and the fact that L approaches L_0 near infinity, the preliminary estimates

$$\|D^2 u\|_p \le C(\|L_t u\|_p + \|u\|_{p,K})$$

with C independent of t and K a fixed compact set determined by the coefficients of L. Then, by Sobolev embedding Theorems, we proved that, when $1 < p < \frac{N}{2}$ the homogeneous estimates hold. The continuity method and the surjectivity of L_0 gave the invertibility of L. The case $p \ge \frac{N}{2}$ required some more efforts. Here we used the Fredholm operators theory to deduce that the index of L_t coincides with the index of L_0 which is zero.

We point out that Nirenberg and Walker ([6, Theorem 3.1]), under assumptions similar to ours, proved some weighted a-priori estimates for elliptic operators. In particular they showed that estimates of the form

$$\|(1+|x|^2)D^2 u\|_p \le C(\|(1+|x|^2)Lu\|_p + \|u\|_p)$$

hold. It can easily proved that these estimates are equivalent to

$$\|D^2 u\|_p \le C\left(\|Lu\|_p + \left\|\frac{1}{(1+|x|^2)}u\right\|_p\right). \tag{2}$$

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