Riesz transforms of some parabolic operators

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Harmonic analysis of Schrödinger operators $A = -\Delta + V(x)$ has attracted attention in recent years. For example, the theory of Hardy and BMO spaces associated to such operators (see [4], [6] and the references there), L^p-boundedness of the associated Riesz transforms $\nabla A^{-1/2}$ (see e.g. [7] or [10]), spectral multipliers ([3]) have been developed. Related operators to Riesz transforms such as $D^2(-\Delta + V)^{-1}$, $V(-\Delta + V)^{-1}$, $V^{\frac{1}{2}}(-\Delta + V)^{-\frac{1}{2}}$, $\nabla(-\Delta + V)^{-\frac{1}{2}}$ have been studied on L^p-spaces under suitable assumptions on the potential V (see [9] or [1]). Less investigated is the L^p-boundedness of the analogous operators associated with parabolic Schrödinger operators $\mathcal{A} = \partial_t - \Delta + V(t,x)$. We refer for example to [2] where the L^p-boundedness of $\nabla^2(\partial_t - \Delta + V(x,t))^{-1}$, or equivalently $V(\partial_t - \Delta + V(t,x))^{-1}$ is proved for a special class of potentials. See also [5] for the case where V is time independent. To our best knowledge, Riesz transforms of $\mathcal{A}$ have not been studied. The aim of our investigation has been to close this gap. We proved that, under suitable assumptions on the potential $V = V(t,.)$, $\nabla\mathcal{A}^{-1/2}$ is bounded on $L^p([0,T]\times\mathbb{R}^N)$ for suitable p.

Boundedness of the operator $\nabla(-\Delta + V(x))^{-1/2}$ on $L^p(\mathbb{R}^N)$ relies heavily on heat kernel bounds, i.e., bounds for the integral kernel of the semigroup $e^{-t(-\Delta+V)}$. For non-negative V such bounds are Gaussian and follow easily from the domination by the Gaussian semigroup. When dealing with Riesz transforms of parabolic operators, $\mathcal{A}$ looks like a degenerate operator in $N+1$ variables $(t, x_1, \cdots, x_N)$ (we do not have ∂_t^2 in the expression of $\mathcal{A}$). Therefore the methods to study $L^p(\mathbb{R}^N)$-boundedness of $\nabla(-\Delta+V(x))^{-1/2}$ do not work for $\nabla(\partial_t - \Delta + V(t,x))^{-1/2}$ on $L^p([0,T]\times\mathbb{R}^N)$. Even in the case $p = 2$ it is not clear (at least to us) whether the latter operator is always bounded. In the case where $V(t,x) = V(x)$ we shall see that the operator $\nabla(\partial_t - \Delta + V(x))^{-1/2}$ is bounded on $L^2([0,T]\times\mathbb{R}^N)$.

Our strategy to prove boundedness of $\nabla(\partial_t - \Delta+V(t,x))^{-1/2}$ on $L^p([0,T]\times\mathbb{R}^N)$ is based on the maximal regularity property of the corresponding non-autonomous Cauchy problem

$$\partial_t u - \Delta u + V(t,.)u = f(t,.),\ u(0) = 0 \ \text{(NACP)}$$

for initial data $f \in L^p([0,T]\times\mathbb{R}^N)$. Indeed the maximal regularity of (NACP) implies that the domain of $\mathcal{A}$ is contained in the domain of $A = -\Delta + V(t,x)$ (but seen as an operator on $L^p([0,T]\times\mathbb{R}^N)$). Combining this embedding with the isomorphism between interpolation spaces and domains of fractional powers will allow us to use the boundedness of Riesz transforms of $-\Delta+V$. This simple idea is quite effective but has a disadvantage in the sense that it gives boundedness of $\nabla(I+\mathcal{A})^{-1/2}$ rather than $\nabla\mathcal{A}^{-1/2}$. If we assume that $V(t,x) \geq c > 0$, then boundedness of $\nabla\mathcal{A}^{-1/2}$ is equivalent to boundedness of $\nabla(I+\mathcal{A})^{-1/2}$.

One of our results asserts the following: suppose that there exists $W \in L^\infty_{loc}(\mathbb{R}^N)$ such that

$$c_1 W(x) \leq V(t,x) \leq c_2 W(x)$$

(a.e.$x \in \mathbb{R}^N$) and all $t \in [0,T]$, and there exists $\beta > 1/2$ such that

$$|V(t,x) - V(s,x)| \leq c_2 W(x)|t-s|^\beta$$

(a.e. $x \in \mathbb{R}^N$) and all $t, s \in [0,T]$, then $\nabla(I+\mathcal{A})^{-1/2}$ is bounded on $L^p([0,T]\times\mathbb{R}^N)$ for all $p \in (1,2]$. If $N \geq 3$ and $W \in L^{N/2-\epsilon}(\mathbb{R}^N) \cap L^{N/2+\epsilon}(\mathbb{R}^N)$ for some $\epsilon > 0$, then $\nabla(I+\mathcal{A})^{-1/2}$ is bounded on $L^p([0,T]\times\mathbb{R}^N)$ for $p \in (2,N)$.

Note that the maximal regularity of (NACP) we need in order to prove this result was studied in [8].

The ideas presented above work also for other operators as elliptic operators with time dependent coefficients. However we concentrated on Schrödinger operators with time dependent potentials.

We finally mention that the boundedness of the Riesz transforms of $\mathcal{A}$ implies that the solution $u(t,x)$ of the Cauchy problem (NACP) satisfies $u \in W^{1,p}_x([0,T]\times\mathbb{R}^N)$. The maximal regularity says that $u \in W^{1,p}_t([0,T]\times\mathbb{R}^N)$. Here $W^{1,p}_y$ denotes the Sobolev space with respect to the variable $y = t$ or $y = x$.

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