Elliptic operators with unbounded diffusion coefficients

G.Metafune [1] , C.Spina [1] and C. Tacelli [1]

[1]Dipartimento di matematica e Fisica, Università del Salento, Italy

The object of our investigation has been the study of existence, uniqueness, qualitative behaviour of solutions of second order elliptic and parabolic problems associated with operators whose coefficients can be unbounded or degenerate in some points. In particular we studied some elliptic operators with unbounded diffusion and drift coefficients of the form

$$(s+|x|^\alpha)\Delta + c|x|^{\alpha-1}\frac{x}{|x|}\cdot\nabla,\ 0\le s\le 1.$$

We first considered the purely second order operator

$$L=(1+|x|^\alpha)\Delta,$$

with $\alpha>0$, in $L^p(\mathbb{R}^N,dx)$, and the related problems

$$u_t - Lu = 0 \qquad u(0)=f;$$

$$\lambda u - \Delta u = f, \qquad \lambda>0.$$

Our analysis was inspired by a previous work due to Fornaro and Lorenzi [1] where the authors proved that, if $\alpha\le 2$, then L generates an analytic semigroup in L^p, for $1\le p\le\infty$. Moreover, if $1<p<\infty$, then

$$D(L_p)=\{u\in L^p : a^{1/2}\nabla u, aD^2u\in L^p\}$$

where $a(x)=(1+|x|^\alpha)$. Actually the authors proved the same results for more general a satisfying $a\ge\delta>0$ and $|\nabla a|\le Ca^{1/2}$.
On the other hand some a-priori estimate for this operators were known also for certain values of α greater then 2. Indeed P. Kree, B. Muckenhoupt and R. Wheeden ([2] and [7]) proved that

$$\||x|^\alpha D^2u\|_p\le C\||x|^\alpha\Delta u\|_p$$

$$\|(1+|x|^\alpha)D^2u\|_p\le C\|(1+|x|^\alpha)\Delta u\|_p$$

for $0<\alpha<\frac{N}{p'}$ where p' is the conjugate exponent of p. Therefore we were expecting for a good characterization of the domain for these values of α.
We studied the case $\alpha>2$ in [3] where it is shown that L generates a semigroup in L^p if and only if $p>N/(N-2)$. The semigroup is always analytic but it is contractive if and only if p is greater than or equal to $(N+\alpha-2)/(N-2)$. According to the above a-priori estimates we gave an explicite description of the domain for $\alpha<\frac{N}{p'}$. We proved also that the resolvent of L is compact in L^p and the spectrum consists of eigenvalues independent of p.
We briefly outiline the main steps to obtain the above results. Set

$$D_{p,max}(L)=\{u\in W^{2,p} : (1+|x|^\alpha)\Delta u\in L^p\}$$

the maximal domain of L and $\hat{L}_p=(L,\hat{D}_p)$ any domain $\hat{D}_p$ contained in the maximal domain, we first proved that

(i) if $N=1,2$ and $1\le p\le\infty$

(ii) or $N\ge 3$ and $p\le N/(N-2)$

then $\rho(\hat{L}_p)\cap[0,\infty[=\emptyset$. We therefore focused on the case $N\ge 3$, $p>N/(N-2)$.
We observed that the equation $-Lu=f$ is equivalent to $-\Delta u=f/(1+|x|^\alpha)$ and then we studied the integral operator

$$Tf(x)=C_N\int_{\mathbb{R}^N}\frac{f(y)\,dy}{(1+|y|^\alpha)|x-y|^{N-2}}$$

and its gradient

$$Sf(x)=C_N(N-2)\int_{\mathbb{R}^N}\frac{f(y)(y-x)\,dy}{(1+|y|^\alpha)|x-y|^{N}}$$

The invertibility of L and many other properties followed from some weighted estimates of T and its gradient.
By using the maximum principle, we deduced that, since L is invertible, then $\lambda-L$ is invertible for $\lambda\ge 0$ and

$$(\lambda-L)^{-1}f\le(-L)^{-1}f$$

for $f\ge 0$.
In such a way we proved that, if $\alpha>2$, $N/(N-2)<p<\infty$ and $\lambda\ge 0$, the operator $\lambda-L$ is invertible on $D_{p,max}(L)$ and its inverse is a positive operator. Moreover

$$\|(\lambda-L)^{-1}\|\le\|T\|.$$

The estimate above only shows that the resolvent is bounded on $[0,\infty[$. This was not sufficient to

apply the Hille-Yosida Theorem in order to prove some results for the parabolic problem. By using variational methods and some Hardy-type inequalities, we proved that, for some values of α (or p), L is a sectorial operator. The constant in the Hardy inequality led to the limitation on α. So we deduced, as partial result, that, if $N \geq 3$, $p > N/(N-2)$, $2 < \alpha \leq (p-1)(N-2)$ (or $p \geq (N+\alpha-2)/(N-2)$), then $(L, D_{p,max}(L))$ generates a positive contractions semigroup in L^p. If $\alpha < (p-1)(N-2)$ (or $p > (N+\alpha-2)/(N-2)$), the semigroup is also analytic.
This has been the starting point to prove, by applying a long iterative process and rescaling techniques, generation of analytic semigroups for every $p > N/(N-2)$.
Concerning the domain characterization, by the weighted estimates of the operator T and by applying the classical Calderon-Zygmund estimate to $(1+|x|^\alpha)u$ we deduced that, if $\alpha < N/p'$, then

$$D_{p,max}(L) = D_p = \{u \in L^p(\mathbb{R}^N) : \\ (1+|x|^{\alpha-2})u, (1+|x|^{\alpha-1})\nabla u, \\ (1+|x|^\alpha)D^2u \in L^p(\mathbb{R}^N)\}.$$

When $\alpha \geq N/p'$, D_p is a proper subset of $D_{p,max}(L)$.
In [4] and [8], by using Kohn-Nirenberg's inequalities and ultracontractivity techniques due to Wang and Bakry, we proved some kernel estimates and we studied the asymptotic behaviour of eigenvalues and eigenfunctions.

Next we considered the whole operators

$$L = (1+|x|^\alpha)\Delta + c|x|^{\alpha-1}(x/|x|)\nabla,$$

as before in L^p spaces with respect to the Lebesgue measure. If α is less or equal than 2, the operator belongs to the class of operators studied in [1]. If $\alpha > 2$, the drift term is not a small perturbation of the principal part and therefore the operator L cannot be studied with perturbative methods starting from the case $c = 0$. Instead, we gave estimates of the fundamental solution, depending on the parameter c, in order to prove solvability. Also in this case we proved that, if $c > 2-N$, L generates a semigroup in L^p if and only if $p > N/(N-2+c)$. The semigroup is always analytic but it is contractive if and only if p is greater than or equal to $(N+\alpha-2)/(N-2+c)$. The description of the domain is given for α smaller than $N/p'+c$. Some kernel estimates are proved as before, by using Kohn-Nirenberg's inequalities and ultracontractivity techniques.

The singular case $s = 0$ without drift (i.e. $c = 0$) is treated in [5], where it is proved that L generates an analytic semigroup in L^p for values of p depending on the parameter α.
Since the operator is degenerate both at 0 and ∞, we studied separately the operators $L_1 = |x|^\alpha\Delta$ in the ball B_R and $L_2 = |x|^\alpha\Delta$ in the exterior domain B_R^c, both with Dirichlet boundary conditions.
Concerning the operator L_2, we observed that it can be treated as the operator $(1+|x|^\alpha)\Delta$ in the whole space $\mathbb{R}^N$. Generation results and domain description for this last operator were already known by [3] in the case $\alpha > 2$ and by [1] in the case $\alpha \leq 2$. It followed that L_2 generates an analytic semigroup for $1 < p < \infty$ when $\alpha \leq 2$ and for $\frac{N}{N-2} < p < \infty$ when $\alpha > 2$, the restriction on p being sharp.
The operator L_1 is singular near the origin. However a generalization of the results of [1] allowed to prove generation of analytic semigroup when $\alpha \geq 2$, together with an explicit desciption of the domain.
The case $\alpha < 2$ required several steps. We first proved that L_1 is invertible and that its resolvent is positive. Then the bound on the resolvent norm $\|(\lambda - L_1)^{-1}\| \leq \|L_1^{-1}\|$ followed for $\lambda > 0$. This however was not enough to obtain generation results by the classical Hille-Yosida Theorem. The operator $L_1 = |x|^\alpha\Delta$ is similar in $L^{\frac{2N}{N-2}}$, via the Kelvin transform, to the operator $|x|^{4-\alpha}\Delta$ defined on the exterior domain B_R^c. Since the operator $|x|^{4-\alpha}\Delta$ generates an analytic semigroup in $L^p(B_R^c)$, $p = (2N)/(N-2)$, consequently, L_1 generates an analytic semigroup in $L^p(B_R)$ for the same p. By interpolation we deduced analiticity for $p \geq \frac{2N}{N-2}$. To conclude, an extrapolation procedure based on the boundedness of the resovent, scaling arguments and the generation results for large p, allowed to prove generation for every $p > \frac{N}{N-\alpha}$. We point out that the above restriction on p is sharp.
Glueing togehther the resolvents of L_1 and L_2 we obtained the results for L.
Moreover, by applying techniques similar to those working for the non-singular case, estimates for the parabolic kernel are obtained.

As further development we are planning to replace the Laplacian with an elliptic operator of the second order in the principal part. The solution of this problem does not seem to be immediate, neither for α less than or equal to 2. Indeed the techniques applied in [1] are based on homogeneus Calderon-Zygmund inequalities that do not hold for more general pure second order elliptic operators.

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