Nonlinear Schrödinger systems with nonzero boundary conditions

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The nonlinear Schrödinger (NLS) equation is a universal model for the behavior of weakly nonlinear, quasi-monochromatic wave packets, and they arise in a variety of physical settings. In particular, the so-called defocusing NLS equation:

$$iq_t + q_{xx} - 2|q|^2 q = 0 \qquad (1)$$

describes the stable propagation of an electromagnetic beam in (cubic) nonlinear media with normal dispersion, and has been the subject of renewed applicative interest in the framework of recent experimental observations in Bose-Einstein condensates and dispersive shock waves in optical fibers.

The defocusing NLS, admits soliton solutions with nonzero boundary conditions (NZBCs), so-called dark/gray solitons, which have the form:

$$q(x,t) = q_0 e^{-2iq_0^2 t}\left[\cos\alpha + i\sin\alpha\,\tanh\xi\right] \qquad (2)$$
$$\xi = q_0 \sin\alpha\,(x + 2q_0\,t\cos\alpha - x_0)$$

with q_0, α and x_0 arbitrary real parameters. Dark soliton solutions are such that $|q(x,t)| \to q_0$ as $x \to \pm\infty$, and appear as localized dips of intensity $q_0^2 \sin^2\alpha$ on the background field q_0.

Even though the inverse scattering transform (IST) as a method to solve the initial-value problem for the NLS equation was first proposed almost 40 years ago [1], with boundary conditions (BCs) taken as

$$q(x,t) \to q_\pm(t) = q_0 e^{-2iq_0^2 t + i\theta_\pm} \quad \text{as} \quad x \to \pm\infty\,,$$

and has been subsequently studied by several authors [2–6], many important issues still remain to be clarified. The main reason why, in spite of the deep experimental relevance of the problem, a complete and rigorous IST theory is still unavailable is due to the fact that one has to deal with solutions that do not decay at space infinity. This makes the IST significantly more involved than in the case of decaying potentials, in particular as far as the analyticity of the eigenfunctions of the associated scattering problem. As a matter of fact, (the analog of) Schwartz class is usually assumed for the potential, which is clearly unnecessarily restrictive. In [2] the issue of establishing the analyticity of the eigenfunctions was addressed by reformulating the scattering problem in terms of a so-called energy dependent potential, but the drawback of that approach is a very complicated dependence of eigenfunctions and data on the scattering parameter.

A most recent step forward in the direction of a rigorous IST for the defocusing NLS with NZBCs was proposed in [8], where we proved that the direct scattering problem is well defined for potentials q such that $q - q_\pm \in L^{1,2}(\mathbb{R}^\pm)$, $L^{1,s}(\mathbb{R})$ being the complex Banach space of all measurable functions $f(x)$ for which $(1+|x|)^s f(x)$ is integrable, and analyticity of eigenfunctions and scattering data was proved.

An important open issue is whether an area theorem can be established, to relate the existence and location of discrete eigenvalues of the scattering problem to the area of the initial profile of the solution, suitably defined to take into account the NZBCs. In [8] we proved that discrete eigenvalues of the scattering problem, if they exist, are confined to two real semi-intervals of $\pm q_0$ whose sizes decrease to zero with the "area"

$$\int_{-\infty}^{0} dx|q(x) - q_-| + \int_0^{+\infty} dx|q(x) - q_+|.$$

As to the inverse problem, we formulated and solved it both via Marchenko integral equations, and as a Riemann-Hilbert problem in terms of a suitable uniform variable. We determined the asymptotic behavior of the scattering data and showed that the linear system solving the inverse problem is well defined. Finally, we developed the triplet method as a tool to obtain explicit multisoliton solutions by solving the Marchenko integral equation via separation of variables.

These results will be relevant in the context of recent theoretical studies and experimental observations of defocusing NLS in the framework of dispersive shock waves in optical fibers [see, for instance, [7] regarding the appearance and evolution of dispersive shock waves when an input (reflectionless) pulse containing a large number of dark or gray solitons is injected in the fiber]. Moreover, this work will pave the way for generalizing similar results to the defocusing vector NLS equation.

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