Parametric equilibrium problems governed by topologically pseudomonotone bifunctions

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Several problems such as Nash economic equilibrium problems, variational inequality problems, equilibrium problems and Kirszbraun problem can be studied as particular cases of the following problem. Let D, E be nonempty sets and consider the function $f : D \times E \to \mathbb{R}$. One needs to find an element $a \in D$ such that $f(a,b) \geq 0 \ \forall b \in E$. We are interested to study a certain stability of solutions when D, E, and f are perturbed.

Let (X, σ), (Y, s) be Hausdorff topological spaces. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions such that $f_n : X \times Y \to \mathbb{R}$ and let $(D_n)_{n\in\mathbb{N}}$, $(E_n)_{n\in\mathbb{N}}$ be such that D_n, E_n are nonempty subsets of X and Y respectively.

For a given $n \in \mathbb{N}$ we consider the following problem:

$(P)_n$ Find an element $a_n \in D_n$ such that

$$f_n(a_n, b) \geq 0 \ \forall b \in E_n. \tag{0.1}$$

Denote by $S(n)$ the set of the solutions for a fixed n. Suppose that $S(n) \neq \emptyset$, for all $n \in \mathbb{N}$.

Along with the problems above we consider the so-called "limit" problem:

(P) Find an element $a \in D$ such that

$$f(a, b) \geq 0 \ \forall b \in E. \tag{0.2}$$

Denote by $S(\infty)$ the set of the solutions for problem (P). We are going to study the following issue: how "far" from f_n the "limit" function f can be so that for any sequence $(a_n)_{n\in\mathbb{N}}$, $a_n \in S(n)$, its cluster points belong to $S(\infty)$. Pointwise convergence for the sequence $(f_n)_{n\in\mathbb{N}}$ is known as having a bad behavior, while continuous convergence should have a better one. Actually, we focus on a certain property of f rather than the properties of $(f_n)_{n\in\mathbb{N}}$.

Our mathematic tools are related to parametric domains, a topological property of a real bifunction, and a condition that relates parametric functions with the "limit" function (of the "limit" problem).

There are numerous problems in nonlinear analysis, like scalar and/or vector equilibrium problems, scalar and/or vector variational inequality problems, where parametric domains appear. For the parametric domains in $(P)_n$ we shall use Mosco's convergence. In the following, besides the topology σ, we also consider a stronger topology τ on X.

Let D, D_n, $n \in \mathbb{N}$ be nonempty sets in X. We shall use the following notations:

$$\tau - D_n = \{a \in X \mid \exists (a_n), a_n \in D_n, \forall n \in \mathbb{N}, a_n \xrightarrow{\tau} a\};$$

$$\tau - D_n = \{a \in X \mid \exists (n_k), \exists (a_{n_k}), a_{n_k} \in D_{n_k}, \forall k \in \mathbb{N}, a_{n_k} \xrightarrow{\tau} a\};$$

$$\sigma - D_n = \{a \in X \mid \exists (n_k), \exists (a_{n_k}), a_{n_k} \in D_{n_k}, \forall k \in \mathbb{N}, a_{n_k} \xrightarrow{\sigma} a\}.$$

Let us recall two kinds of set convergence.

(PK) The sequence $(D_n)_{n\in\mathbb{N}}$ is said to converges in the Painlevé-Kuratowski sense to D and notes $D_n \xrightarrow{PK} D$ if

$$\tau - D_n \subseteq D \subseteq \tau - D_n;$$

(M) The sequence $(D_n)_{n\in\mathbb{N}}$ is said to converges in the Mosco sense to D and notes $D_n \xrightarrow{M} D$ if

$$\sigma - D_n \subseteq D \subseteq \tau - D_n.$$

It is clear that if $D_n \xrightarrow{M} D$ then $D_n \xrightarrow{PK} D$. If X is a normed space and σ is chosen as the weak topology and τ as the norm topology, then obviously, $D_n \xrightarrow{M} D$ implies $D_n \xrightarrow{PK} D$ as $n \to \infty$.

Equivalently, one has $D_n \xrightarrow{M} D$ as $n \to \infty$ if:

(a) for every subsequence $(a_{n_k})_{k\in\mathbb{N}}$ with $a_{n_k} \in D_{n_k}$, and $a_{n_k} \xrightarrow{\sigma} a$ imply $a \in D$;

(b) for every $a \in D$, there exists $(a_n)_{n\in\mathbb{N}}$, $a_n \in D_n$ such that $a_n \xrightarrow{\tau} a$.

A function $f : X \times X \to \mathbb{R}$ is called topological pseudomonotone provided $\forall (a_n)$, $a_n \in X$, $a_n \xrightarrow{\sigma} a$ such that $\liminf_{n\to\infty} f(a_n, a) \geq 0$ one has $\limsup_{n\to\infty} f(a_n, b) \leq f(a, b)$ for all $b \in X$. The *classical* notion of topological pseudomonotonicity was introduced by Brézis for nonlinear differential operators. We shall give a slight generalization of this notion.

We say that a function $f : X \times Y \to \mathbb{R}$ is top-pseudomonotone if $\forall (a_n)_{n\in\mathbb{N}}$, $a_n \in X$, $a_n \xrightarrow{\sigma} a$ such that $\liminf_{n\to\infty} f(a_n, b^*) \geq 0$ for an element $b^* \in Y$ one has

$$\limsup_{n\to\infty} f(a_n, b) \leq f(a, b) \ \forall b \in Y.$$

If $f : X \times X \to \mathbb{R}$ is topologically pseudomonotone then it is also top-pseudomonotone.

In order to state our main result, we deal with the following condition for $(f_n)_{n\in\mathbb{N}}$:

(**C**) For each subsequence $(a_{n_k})_{k\in\mathbb{N}}$, $a_{n_k} \in S(n)$ if $a_{n_k} \xrightarrow{\sigma} a$, $b_{n_k} \in E_{n_k}$, $b \in E$, and $b_{n_k} \xrightarrow{s} b$, then

$$\liminf_{k\to\infty} \left(f_{n_k}(a_{n_k}, b_{n_k}) - f(a_{n_k}, b) \right) \leq 0.$$

There are some situations when this condition applies. The most ideal can be considered when f_n, $n \in \mathbb{N}$ are continue and $(f_n)_{n\in\mathbb{N}}$ uniformly converges to f.

One of the our results is the following

Theorem 1 *Let X be a Hausdorff topological space with σ and τ two topologies on X, $\sigma \subseteq \tau$. Let (Y, s) be a Hausdorff topological space. Let D_n, E_n ($n \in \mathbb{N}$) be nonempty subsets of X and Y respectively. Suppose that $S(n) \neq \emptyset$, for each $n \in \mathbb{N}$, and the following conditions hold:*

i) $\sigma - D_n \subseteq D$ *and* $E \subseteq s - E_n$;

ii) $f_n(\cdot, \cdot)$ $(n \in \mathbb{N})$ *verify condition* (**C**);

iii) $f(\cdot, \cdot) : X \times Y \to \mathbb{R}$ *is top-pseudomonotone.*

Then $\sigma - S(n) \subseteq S(\infty)$, *i.e. for each subsequence of solutions to* $(P)_n$, $a_{n_k} \xrightarrow{\sigma} a$ *imply* $a \in S(\infty)$.

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