Some questions on plane curves

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In previous paper ([1]) a new notion of tangency is formulated in suitable metric spaces and an equivalent metric formulation of the notion of tangency at a point of the graphic of a real function ([1] (Example 2.6) and [2](Teorema 1.1)) is considered. Then it is possible to reconsider the notion of tangent line at a point of a plane curve (and also of normal line) in a very different way with respect the classical one. This paper is devoted to consider such notion and to investigate some new aspects and questions on plane curves.

It is well know that ”... Not all curves are rectificable; some do not have a tangent at any of their points...” (see[3] (pag.46)). From your study at every point of a plane curve there is, in a some specified meaning, at least a ”tangent direction” and it is possible, from a pure theoretical point of view, to have many ”‘tangent directions”’ at every point. This new point of view give some problems wich seem news and interesting.

Following [1], we consider two abstract operations. Let (X, d) be a metric space and A, B be non-empty, compact (or locally compact) subsets of X. Assume that $x_0 \in A \cap B$ is an accumulation point of A. We define the functions:

$$\underline{D}_{x_0}(A,B) = \liminf_{A\setminus\{x_0\}\ni x\to x_0} \frac{d(x,B)}{d(x,x_0)}; \tag{0.1}$$

$$\overline{D}_{x_0}(A,B) = \limsup_{A\setminus\{x_0\}\ni x\to x_0} \frac{d(x,B)}{d(x,x_0)}; \tag{0.2}$$

where $d(x, B) = \inf\{d(x, y)|\ y \in B\}$.

When $\underline{D}_{x_0}(A,B) = \overline{D}_{x_0}(A,B)$, we write $D_{x_0}(A,B)$. We remark that: $0 \leq \underline{D}_{x_0}(A,B) \leq \overline{D}_{x_0}(A,B) \leq 1$; hence we have: $\overline{D}_{x_0}(A,B) = 0 \Longrightarrow \underline{D}_{x_0}(A,B) = D_{x_0}(A,B) = 0$. The previous operations are investigated, early and with other motivations, from different authors (see [4] [5], [6], [7],[8],[9], [10], [11]).

In [2] we prove the following result, establishing the connession with the usual notion of tangent line. Take as metric space $\mathbb{R}^2$, endowed with the usual euclidean metric, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and $G_f \subset \mathbb{R}^2$ its graph. Let $p_0 = (x_0, f(x_0)) \in G_f$, and consider the functions $\underline{D}_{p_0}$, $\overline{D}_{p_0}$. For r a straight line through p_0, the following conditions are equivalent:

(i) r is the tangent line to G_f at p_0;

(ii) $\overline{D}_{p_0}(r, G_f) = 0$ and $\underline{D}_{p_0}(s, G_f) > 0$ for every line $s \neq r$ through p_0;

(iii) $\overline{D}_{p_0}(G_f, r) = 0$ and $\underline{D}_{p_0}(G_f, s) > 0$ for every line $s \neq r$ through p_0.

Then one can agree the following definition:

Let A, B be non-empty, compact (or locally compact) sets of the metric space X and let x_0 be an accumulation point of A and B. We say that A *is tangent to* B *in* x_0 if and only if $D_{x_0}(A,B) = 0$.

We say that A, B *are tangent in* x_0 if and only if both $D_{x_0}(A,B)$ and $D_{x_0}(B,A)$ exist and $D_{x_0}(A,B) = D_{x_0}(B,A) = 0$.

We remark that if A is tangent to B or A, B are tangent in x_0 with respect to the metric d, then the same occur with respect every metric d_1 equivalent to d.

Now we consider the metric space (X, d), where $X = \mathbb{R}^2$ and d is the usual euclidean metric; we denote with $\mathcal{R}(x)$ the set of half-straight lines through the point x, that is: $r \in \mathcal{R}(x) \Longleftrightarrow \exists v \in \mathbb{R}^2,\quad ||v|| = 1,\quad r = \{x + tv | t \geq 0\}$.

The following condition hold:

$$\forall r \in \mathcal{R}(x) \Longrightarrow x \in r; \tag{0.3}$$

$$\forall r \in \mathcal{R}(x), \textit{every bounded sequence in } r \textit{ is compact}; \tag{0.4}$$

$$\forall r, s \in \mathcal{R}(x) \Longrightarrow \exists D_x(r,s), \exists D_x(s,r) \textit{ and } D_x(r,s) = D_x(s,r); \tag{0.5}$$

$$\forall r, s \in \mathcal{R}(x): \quad D_x(r,s) = 0 \Longrightarrow r = s; \tag{0.6}$$

$$0 < d(x,y) \Longrightarrow \exists r \in \mathcal{R}(x), \quad \exists s \in \mathcal{R}(y) : y \in r \quad \textit{and} \quad x \in s; \tag{0.7}$$

$$\forall x \in \mathbb{R}^2, \forall (r_n), r_n \in \mathcal{R}(x) : \exists (r_{n_k}) \subseteq (r_n), \ \exists s \in \mathcal{R}(x) \textit{ such that } \lim_n D_x(r_{n_k}, s) = 0. \tag{0.8}$$

We will restrict the considerations on a given simple plane curve. Now, let $\gamma : [0,1] \to \mathbb{R}^2$ be continuous and iniective, denote $\Gamma = \{\gamma(t) | t \in [0,1]\}$ and $x_0 = \gamma(t_0), t_0 \in]0,1[$.

We have

$$\exists r_1 \in \mathcal{R}(x_0) \text{ such that } \underline{D}_{x_0}(r_1, \Gamma) = \min\{\underline{D}_{x_0}(s, \Gamma) | s \in \mathcal{R}(x_0)\} \tag{0.9}$$

$$\exists r_2 \in \mathcal{R}(x_0) \text{ such that } \underline{D}_{x_0}(r_2, \Gamma) = \max\{\underline{D}_{x_0}(s, \Gamma) | s \in \mathcal{R}(x_0)\} \tag{0.10}$$

$$\exists r_3 \in \mathcal{R}(x_0) \text{ such that } \overline{D}_{x_0}(r_3, \Gamma) = \min\{\overline{D}_{x_0}(s, \Gamma) | s \in \mathcal{R}(x_0)\} \tag{0.11}$$

$$\exists r_4 \in \mathcal{R}(x_0) \text{ such that } \overline{D}_{x_0}(r_4, \Gamma) = \max\{\overline{D}_{x_0}(s, \Gamma) | s \in \mathcal{R}(x_0)\} \tag{0.12}$$

The elements of $\mathcal{R}(x_0)$ in (0.9), (0.10) can be considered as different types of "'tangent direction"' to the curve γ at the point x_0 and the elements in (0.12) as "'normal direction"'.

In the classical theory of tangency, the tangent line is strictly related to the limit of secant lines to Γ; this can be a justification for the following results:

$$\exists s_1 \in \mathcal{R}(x_0) \text{ such that } \underline{D}_{x_0}(s_1, \Gamma) = \liminf_{\Gamma \setminus x_0 \ni x \to x_0} \underline{D}_{x_0}(r_{xx_0}, \Gamma) \tag{0.13}$$

$$\exists s_2 \in \mathcal{R}(x_0) \text{ such that } \underline{D}_{x_0}(s_2, \Gamma) = \limsup_{\Gamma \setminus x_0 \ni x \to x_0} \underline{D}_{x_0}(r_{xx_0}, \Gamma) \tag{0.14}$$

$$\exists s_3 \in \mathcal{R}(x_0) \text{ such that } \overline{D}_{x_0}(s_3, \Gamma) = \liminf_{\Gamma \setminus x_0 \ni x \to x_0} \overline{D}_{x_0}(r_{xx_0}, \Gamma) \tag{0.15}$$

$$\exists s_4 \in \mathcal{R}(x_0) \text{ such that } \overline{D}_{x_0}(s_4, \Gamma) = \limsup_{\Gamma \setminus x_0 \ni x \to x_0} \overline{D}_{x_0}(r_{xx_0}, \Gamma). \tag{0.16}$$

where r_{xx_0} is the half-straight line with $v = \frac{x - x_0}{||x - x_0||}$

From a theoretical point of view some questions arise in consideration of the obtained results, for example:

-the question to reconsider this point of view with respect to other metric in R^2 not equivalent to euclidean metric, also non homogeneous metric. In such situation it is possible to need a change of the sets $\mathcal{R}(x)$ with some appropriate "'geodetic"' directions.

See also:
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