Desingularizing the dynamics generated by logarithmic potentials

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Typically, problems in celestial mechanics involve the Newtonian gravitational potential, which is singular when two or more bodies (intended as point masses) collide together.

In a seminal paper, R. McGehee introduced a rescaling technique that regularizes the dynamics of n-body gravitational problems. It consists of a polar-type change of coordinates in the configuration space, together with a suitable rescaling of the momentum. This technique may be applied to singular homogeneous potentials of any degree.

However, homogeneous potentials are not the only interesting singular potentials that arise in the applications. Logarithmic potentials appear in several physical scenarios: in models of astrodynamics; in the dynamics of a charged particle in a cylindrically symmetric electric field, and in the mathematical theory of vortex filaments of an ideal fluid. In addition, the logarithmic potential $V(x) = -\log(|x|)$ may be considered a sort of limit case for $\alpha \to 0$ of the homogeneous potentials $V_\alpha(x) = |x|^-\alpha$.

There are very few studies on the regularization of the logarithmic potential, and they all treat the planar case. In a collaboration with the Basque Center for Applied Mathematics, we worked out a McGhee-like technique for the regularization of the one-body logarithmic dynamics on a sphere.

In this setting, a necessary condition in order to have a collision with the singularity, or to reach its antipodal point, is to have zero angular momentum. In other words, the only trajectories that reach the singularity are those that coincide with the meridians, when the singularity is at one of the poles. In effect, when one restricts this problem to non-singular orbits, it is easy to see that the problem is integrable.

We show that in this problem there is much more regularity than the presence of the singularity would allow us to suspect. In fact, the McGehee regularization lets us prove the following theorem, based on the notion of *collision-transmission* trajectories.

Definition 1. *Let (ϕ_V, θ_V) be the coordinates of the point interacting with the singular logarithmic potential and let $\gamma(t) = (\phi, \theta)(t) : [0, T_s) \to \boldsymbol{S}$ be a collision solution ending in the singularity at time T_s. We define as* collision-transmission trajectory *the path $\bar\gamma : [0, 2T_s] \to \boldsymbol{S}$ given by*

$$\bar\gamma(t) := \begin{cases} \gamma(t), & t \in [0, T_s) \\ (\phi_V, \theta_V) & t = T_s \\ (2\phi_V - \phi(2T_s - t), & \\ \qquad 2\theta_V - \theta(2T_s - t)) & t \in (T_s, 2T_s] \end{cases}$$

Based on the previous definition, we prove that:

Theorem 1. *The flow obtained by extending the collision solutions with the collision-transmission trajectory is continuous with respect to initial data anywhere out of the vortex and of the antipodal points.*

In a different paper we return to the plane, but we study a multi-body problem governed by the equation

$$\Gamma_j \ddot{\boldsymbol{q}}_j(\sigma) + \sum_{k \neq j} \frac{\Gamma_k}{2\pi} \frac{\boldsymbol{q}_j(\sigma) - \boldsymbol{q}_k(\sigma)}{\|\boldsymbol{q}_j(\sigma) - \boldsymbol{q}_k(\sigma)\|^2} = 0, \qquad j = 1, \dots N$$

where Γ_j is the strength of the logarithmic potential associated to the $j-$th body.

This problem is amenable to analysis by restricting the investigation to the motions that respect some symmetry constraint. In particular we have studied the case where one body of strength γ is left still at the origin of the coordinate system, and four other bodies having the same strength move along orbits equivariant with respect to the Klein group D_2, which has order 4.

As for the spherical problem, we are able to regularize the total collision that happens when all five bodies collide simultaneously. We are also able to define collision-transmission orbits for the binary collisions that occur in this problem for generic initial conditions. In this way the flow can be defined for arbitrarily long times, and we study the conditions under which the dynamics is bounded or unbounded, that is whether or not there exist orbits that reach infinity.

REFERENCES

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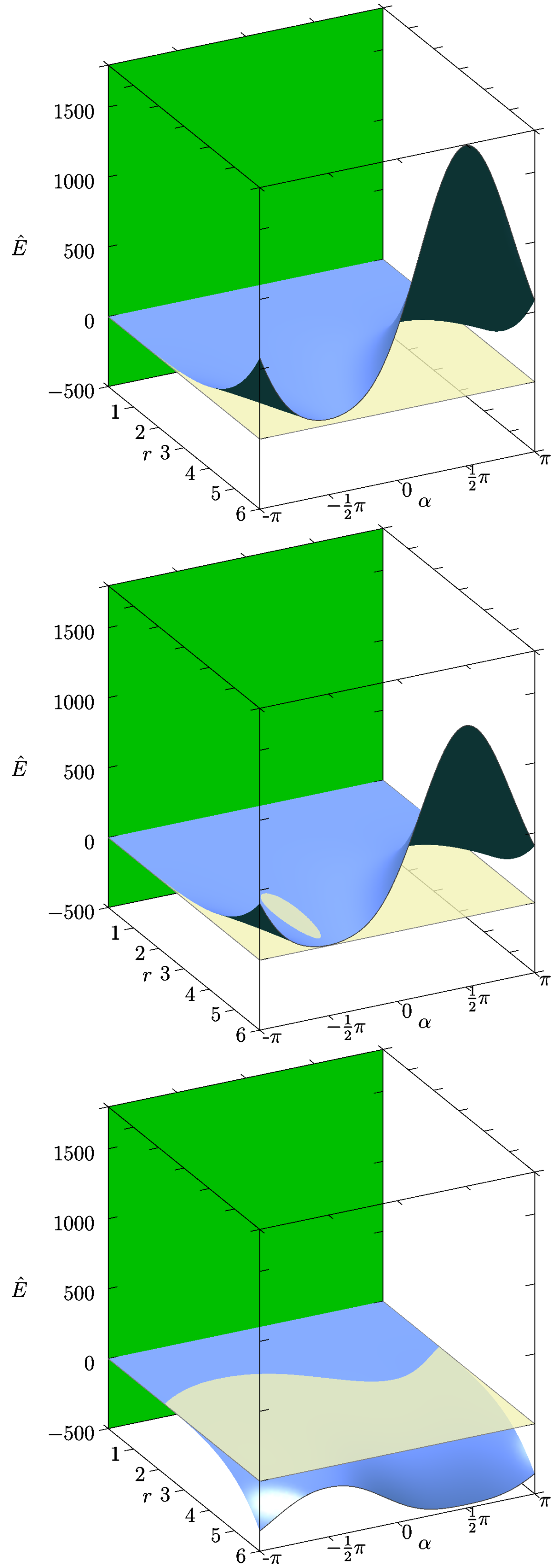

Figure 1. Surfaces of constant kinetic energy in McGehee coordinates for the logarithmic one-body problem on a sphere.