Degenerate elliptic and parabolic operators on domains*

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The study of degenerate elliptic operators started in the fifties and has been the object of many researches in a wide generality, since the seminal works of W. Feller in one dimension and J.J. Kohn and L. Nirenberg in higher dimensions. Of particular interest is the case of degeneracy at the boundary for second-order elliptic operators, and the results heavily depend on the behaviour of the coefficients at the boundary, i.e., on the order and the direction of degeneracy. The case of high order degeneracy is in some sense easier and is treated in [6] and, in a simpler way, in [3]. A challenging borderline case occurs if the diffusion coefficients fully degenerate of first order in the normal direction to the boundary. In this case the *drift term* in normal direction is (roughly speaking) of the same order as the *diffusion part* and the sign and size of the drift coefficients play a crucial role. Here the model problem is given by the operator

$$L = -y\Delta + b_0 \cdot \nabla_x + b\partial_y \tag{1}$$

on the halfspace $\mathbb{R}^n_+ = \{(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} \,|\, y > 0\}$, where $b_0 \in \mathbb{R}^{n-1}$, $b \in \mathbb{R}$ and $\Delta = \Delta_x + \partial_y^2$. It has been shown in [2] that $-L$ with domain $D(L) = \{u \in W^{1,p}(\mathbb{R}^n_+) \,|\, u(\cdot,0) = 0,\ Lu \in L^p(\mathbb{R}^n_+)\}$ generates an analytic semigroup on $L^p(\mathbb{R}^n_+)$ if $b > -1/p$. The techniques of [2] heavily depend on the condition $b > -1/p$, which allows to control gradient terms by the operator L through a Hardy type inequality which seems to fail for $b \le -1/p$ and without the Dirichlet boundary conditions embodied in the domain $D(L)$. In fact, it has been noted in Example 2.11 of [2] that the generation result of this paper breaks down for $b \le -1/p$, already in the one dimensional case. In [5] we study the one dimensional case, i.e., the operator

$$A = -xD^2 + bD, \qquad \text{for all } b \in \mathbb{R}.$$

It turns out that in the case $b \le -1/p$ the behavior of A is rather complex and has several quite unexpected features. We started the analysis working on the interval $(0,1)$ and imposing Dirichlet boundary conditions $u(1) = 0$, throughout. On the interval $(\varepsilon, 1)$, we can equip the operator A with *Dirichlet* ($u(\varepsilon) = 0$) or *Neumann* ($u'(\varepsilon) = 0$) boundary conditions at $x = \varepsilon$, letting finally $\varepsilon \to 0$. Let us define the domains

$$\begin{aligned} D_{p,\max} &= \big\{u \in L^p(0,1) \cap W^{2,p}_{\rm loc}((0,1]) \,\big|\, Au \in L^p(0,1), u(1) = 0\big\}, \\ D_{p,\rm en} &= \left\{u \in D_{p,\max} \,\Big|\, \int_0^1 x\,|u'|^2\,|u|^{p-2}\,dx < \infty\right\}, \\ D_{p,\rm reg} &= \big\{u \in W^{1,p}(0,1) \,\big|\, xu'' \in L^p(0,1),\ u(1) = 0\big\}. \end{aligned}$$

In [5] we show the following result.

Theorem 1. *For every $b \in \mathbb{R}$ and $p \in (1,\infty)$ there exist $\omega_p > 0$ and a subspace $D_p \subseteq D_{p,\max}$ such that $-A_p = (-A, D_p)$ generates a strongly continuous positive semigroup $(T(t))_{t\ge 0}$ with $\|T(t)\| \le e^{-\omega_p t}$ for $t \ge 0$. Moreover, the semigroup $(T(t))_{t\ge0}$ is analytic in $L^p(0,1)$.*

We write $D^0_{p,\rm en}$ and $D^0_{p,\rm reg}$ if we include the condition $\lim_{x\to 0^+} u(x) = 0$ in the respective space. Mainly using the explicit expression for A_p^{-1}, in the same paper we show that

$$\begin{aligned} &D_p = D_{p,\rm reg} = D_{p,\max} = D_{p,\rm en} && \text{if } \ b \le -1 - \tfrac{1}{p}, \\ &D_p = D_{p,\rm reg} = D_{p,\rm en} \ne D_{p,\max} && \text{if } \ -1 - \tfrac{1}{p} < b \le -1, \\ &D_p = D^0_{p,\rm en} = D^0_{p,\max} \supsetneq D^0_{p,\rm reg} && \text{if } \ -1 < b \le -1/p, \\ &D_p = D^0_{p,\rm en} = D^0_{p,\max} = D^0_{p,\rm reg} && \text{if } \ b > -1/p. \end{aligned}$$

There are several striking features: For $b \le -1$ we cannot impose boundary conditions at $x = 0$ in the sense that any restriction of $-A$ to a proper subset of $D_{p,\max}$, respectively $D_{p,\rm en}$, cannot be a generator.

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Further, the regularity contained in the domain varies with b. In particular we loose the $W^{1,p}$ regularity precisely for $b \in (-1, -1/p]$ where we still keep the Dirichlet boundary condition at 0.

To see another surprising fact, let $u_\varepsilon \in W^{2,p}(\varepsilon,1) \cap W_0^{1,p}(\varepsilon,1)$ satisfy $Au_\varepsilon = 1$ and let $b \le -1/p$. Then, the norms $\|u_\varepsilon'\|_p$ of the approximations explode as $\varepsilon \to 0$, although $D_p \subset W^{1,p}(0,1)$ for $b \le -1$.

Motivated by these observations, we also study Neumann approximations of A on $(\varepsilon, 1)$, where we impose $u_\varepsilon'(\varepsilon) = 0$ instead of $u_\varepsilon(\varepsilon) = 0$. It turns out that for $b \le -1$ in the limit we obtain again A_p, but now with an approximation which is stable in $W^{1,p}$. For $b \in (-1, -1/p)$ the approximations are again stable in $W^{1,p}$, and now they give a new generator $-A_{p,N}$ of an analytic semigroup with domain $D(A_{p,N}) = D_{p,\mathrm{reg}}$. Compared to the operator A_p constructed via Dirichlet approximations, we loose the boundary condition $u(0) = 0$, but gain the optimal regularity $u \in W^{1,p}(0,1)$. Summarising, we have:

Theorem 2. *a) For $b < -1/p$, the operator $-A_{p,N} = (-A, D_{p,\mathrm{reg}})$ generates a positive analytic C_0–semigroup $T_N(\cdot)$ on $L^p(0,1)$. For $b \le -1$, the operator $-A_{p,N}$ coincides with the generator $-A_p$ from Theorem 1, and hence $T_N(\cdot) = T(\cdot)$. For $b \in (-1, -1/p)$, the operator $A_{p,N}$ differs from A_p, hence $T_N(\cdot) \ne T(\cdot)$, and we have $D_{p,\mathrm{reg}} \subsetneqq D_{p,\mathrm{en}} = D_{p,\max}$.*
b) If $b \ge -1/p$, there are $f \in L^p(0,1)$ such that $\|u_\varepsilon\|_p \to \infty$ as $\varepsilon \to 0$ for the functions $u_\varepsilon \in D_{p,\varepsilon}^N$ with $Au_\varepsilon = f$. Hence, the Neumann approximation does not work in this case.

In [4] we study elliptic operators L of second order with Dirichlet boundary conditions on a bounded domain Ω whose diffusion coefficients degenerate at $\partial\Omega$ *in tangential directions*. The degeneracy affects only the tangential variables and is of the order of the distance from $\partial\Omega$. The prototype of this class is the well–known Tricomi operator $L = -y\Delta_x - \partial_y^2$ in the upper halfspace $\mathbb{R}_+^n$. In the general case of a bounded domain $\Omega \subset \mathbb{R}^n$, i.e., $\Omega = \{\varrho > 0\}$, $\partial\Omega = \{\varrho = 0\}$, $\nabla\varrho(\xi)$ directed along the inward normal vector if $\xi \in \partial\Omega$ (ϱ is an extension of the distance function to $\partial\Omega$), the operator L is of the form

$$L = -\mathrm{tr}(\mathbf{a} \otimes \mathbf{a}\, D^2) - \varrho \sum_{i,j=1}^n a_{ij}\partial_{ij} - \sum_{i=1}^n b_i \partial_i. \tag{2}$$

Here a_{ij}, b_i are continuous functions, a_{ij} satisfy the ellipticity condition

$$\sum_{i,j=1}^n a_{ij}(\xi)\tau_i\tau_j \ge \mu_0|\tau|^2, \quad \text{for every } \xi \in \partial\Omega,\ \tau \in \mathbb{R}^n \text{ with } \tau\cdot\mathbf{a}(\xi) = 0,$$
$$\sum_{i,j=1}^n \Big(a_i(\xi)a_j(\xi) + \varrho(\xi)a_{ij}(\xi)\Big)\zeta_i\zeta_j \ge \mu(\xi)|\zeta|^2, \quad \text{for every } \xi \in \Omega,\ \zeta \in \mathbb{R}^n$$

(for some constant $\mu_0 > 0$ and a suitable function μ with $\inf_K \mu > 0$, for any compact set K contained in Ω) and the vector field $\mathbf{a}$ is C^2 and non tangential on $\partial\Omega$. Hence, the tangential degeneracy of the diffusion is expressed by the properties of $\mathbf{a}$. We have the following results.

Theorem 3. *Under the above assumptions, the operator $(-L, D_p(L))$ with domain*

$$D_p(L) = \big\{u \in W^{2,p}_{\mathrm{loc}}(\Omega) \cap W_0^{1,p}(\Omega) : \varrho|D^2u|, \mathrm{tr}(\mathbf{a}\otimes\mathbf{a}\, D^2u), \sqrt{\varrho}\,|D^2u\,\mathbf{a}| \in L^p(\Omega)\big\},$$

generates an analytic semigroup in $L^p(\Omega)$ for $p \in (1,\infty)$.

Theorem 4. *Under the above assumptions, the operator $(-L, D_0(L))$ with*

$$D_0(L) = \Big\{u \in C(\overline{\Omega}) \cap \bigcap_{1\le p<\infty} W^{2,p}_{\mathrm{loc}}(\Omega)\,\Big|\,\mathbf{a}\cdot\nabla u,\ \sqrt{\varrho}\,\nabla u,\ Lu \in C(\overline{\Omega}),\ u_{|\partial\Omega} = 0\Big\},$$

generates an analytic semigroup $T(\cdot)$ in $C(\overline{\Omega})$. It further holds

$$\|\mathbf{a}\cdot\nabla u\|_{L^\infty(\Omega)} + \|\sqrt{\varrho}\,\nabla u\|_{L^\infty(\Omega)} \le C|\lambda|^{-\frac12}\|f\|_{L^\infty(\Omega)}$$

for every $Re\,\lambda > \omega_0$, $f \in C(\overline{\Omega})$, $u = (\lambda + L)^{-1}f$, and some $\omega_0 \ge 0$. Moreover, this semigroup is contractive, positive, compact, exponentially stable, and it is the restriction of the semigroups on $L^p(\Omega)$ obtained in Theorem 3.

Thus, we have a complete theory including existence, uniqueness and (maximal) regularity of the elliptic and parabolic problems for L in L^p spaces and in spaces of continuous functions. Moreover, we establish consistency, positivity, compactness and exponential stability of the analytic semigroups generated by L. The domain of L is computed explicitly in L^p, $p \in (1, +\infty)$.

In [1] we study again a class of elliptic operators L that degenerate at the boundary of a bounded open set $\Omega \subset \mathbb{R}^d$ with $\partial\Omega$ of class C^∞, but in a different perspective, i.e., we assume that L possesses a symmetrising invariant measure μ. Such operators are associated with diffusion processes in Ω which are invariant for time reversal. Indeed, the operator

$$Lu = \frac{1}{2}\mathrm{Tr}\,[\sigma\sigma^* D^2 u] + \langle b, Du\rangle, \quad u \in C^2(\Omega)\,, \tag{3}$$

where $b : \overline{\Omega} \to \mathbb{R}^d$ is of class C^1, $\sigma : \overline{\Omega} \to L(\mathbb{R}^d)$ is continuous on $\overline{\Omega}$, of class $C^1(\Omega)$ and such that, setting $a = \sigma\sigma^*$, $\det a(x) > 0\ \forall\, x \in \Omega$, is the Kolmogorov operator associated with the diffusion process described by the stochastic differential equation

$$dX(t) = b(X(t))dt + \sigma(X(t))dW(t), \qquad X(0) = x \in \Omega, \tag{4}$$

where W is a Brownian motion. If there exists $\rho \in C^1(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$ such that $aD\rho + (g - 2b)\rho = 0$, where $g_j = \sum_{i=1}^d D_i a_{ij},\ j = 1, ..., d$, then $L^*\rho = 0$, and $\mu(dx) = \rho(x)dx$ is an invariant measure for X. In this case, the process $Y(t) + X(1-t)$ is a solution of the same SDE, the operator L in (3) is a *gradient system*, it can be written in the form $Lu = \frac{1}{2\rho}\mathrm{div}\,\big(\rho a Du\big)$ and is symmetric in $L^2(\Omega, \mu)$. After showing that the corresponding elliptic equation $\lambda u - Lu = f$ has a unique variational solution for any $\lambda > 0$ and $f \in L^2(\Omega, \mu)$, under further hypotheses we obtain new results for the characterization of the domain of L.

Theorem 5. *Let $Lu = \frac{1}{2}\alpha\Delta u + \langle b, Du\rangle$ and, beside the previous hypotheses, assume the following:*

1. *$\alpha,\ b \in C^\infty(\overline{\Omega}),\ 0 \le \alpha,$ and $\alpha(x) = 0 \iff x \in \partial\Omega$.*
2. *There exists $\rho \in C^\infty(\Omega)$ such that $\rho \in L^1(\Omega)$ and the equation $\alpha(x)D\log\rho(x) + D\alpha(x) = 2b(x)$ holds in Ω.*

Then, the domain of the variational operator L in $L^2(\Omega, \mu)$ can be characterised as follows:

$$D(L) = \{u \in W^{2,2}_{loc}(\Omega) :\ Du \in L^2(\Omega, \mu; \mathbb{R}^d),\ \alpha\Delta u \in L^2(\Omega, \mu)\}. \tag{5}$$

The main point in the above result is the condition $Du \in L^2(\Omega, \mu; \mathbb{R}^d)$, which follows from the estimate $\|Du\|_2 \le c(\lambda)\|f\|_2$ for the solution of $\lambda u - Lu = f$. This last estimate is stronger than the usual one $\|\alpha^{1/2}Du\|_2 \le c(\lambda)\|f\|_2$. In the particular case $b = \frac{k}{2}D\alpha$, $k \ge 1$, $\rho = \alpha^{k-1}$ and less regularity is required to get the above characterisation of the domain, that in this case reads

$$D(L) = \{u \in W^{2,2}_{loc}(\Omega) :\ Du \in L^2(\Omega, \mu; \mathbb{R}^d),\ \alpha\Delta u \in L^2(\Omega, \mu)\}. \tag{6}$$

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