Heat semigroup and geometric measure theory*

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Since its first definition given by E. de Giorgi in 1954 in [10], the notion of *variation* of a function of several real variables (which is called *perimeter* in the relevant case of characteristic functions) appeared to be closely related to the short time behaviour of the heat semigroup. If $u:\mathbb{R}^n\to\mathbb{R}$ is in $L^1(\mathbb{R}^n)$ then its variation is defined as

$$|Du|(\mathbb{R}^n)=\sup\Bigl\{\int_{\mathbb{R}^n} u\,\mathrm{div}\,\phi dx:\ \ \phi\in C^1_c(\mathbb{R}^n,\mathbb{R}^n),\ \ \|\phi\|_\infty\le 1\Bigr\} \tag{1}$$

and the condition $|Du|(\mathbb{R}^n)<+\infty$ is equivalent to saying that the distributional gradient of u is a $\mathbb{R}^n$-valued Radon measure. In this case, we write $Du=\sigma_u|Du|$ and u is of *bounded variation*, $u\in BV(\mathbb{R}^n)$ for short, see [3]. Given $u\in L^1(\mathbb{R}^n)$, denote by W_t the *heat semigroup* in $\mathbb{R}^n$, i.e., the function

$$w(x,t)=W_tu(x)=\frac{1}{(4\pi t)^{n/2}}\int_{\mathbb{R}^n}u(y)\exp\Bigl\{\frac{|x-y|^2}{4t}\Bigr\}dy \tag{2}$$

is the solution of the Cauchy problem in $\mathbb{R}^n\times(0,+\infty)$ $w_t=\Delta w,\ \ w(x,0)=u(x)$. Then, the equality

$$|Du|(\mathbb{R}^n)=\lim_{t\to 0}\int_{\mathbb{R}^n}|\nabla W_tu(x)|dx \tag{3}$$

holds, in the sense that the right hand side in the above formula is finite if and only if $u\in BV(\mathbb{R}^n)$. Another interesting connection between the variation of a function and the heat semigroup is shown by the formula

$$|Du|(\mathbb{R}^n)=\lim_{t\to 0}\frac{\sqrt{\pi}}{2\sqrt{t}}\int_{\mathbb{R}^n\times\mathbb{R}^n}|u(x)-u(y)|\exp\Bigl\{\frac{|x-y|^2}{4t}\Bigr\}dxdy, \tag{4}$$

which for u characteristic function has been pointed out by M. Ledoux in 1994 in [12], in connection with the *isoperimetric inequality*. Both these connections between heat semigroup and the variation of a function have been deepened in the last few years in various contexts, such as Riemannian manifolds [9], [14], subsets of Euclidean spaces [7] and Wiener spaces, i.e., (infinite dimensional) Banach or Hilbert spaces endowed with suitable differential structures and Gaussian measures. In this last framework, which is strongly related to stochastic analysis, the heat semigroup is replaced by the *Ornstein-Uhlenbeck semigroup* whose invariant measure is the given Gaussian measure. In this sense, it appears to be the most natural semigroup to consider. In these situations all the quantities can be introduced through intrinsic variants of (1), (2), but relations (3), (4) do not follow from the Euclidean case.

In the paper [8] the validity of (3), (4) is addressed in the case of *Carnot groups*. A Carnot group $\mathbb{G}$ (named after Nicolas Léonard Sadi Carnot because of the connections of such an algebraic structure with the rational thermodynamics) is $\mathbb{R}^n$ endowed with a group operation $\circ$ and a dilation operation $D(\lambda)x=(\lambda^{\omega_1}x_1,\dots,\lambda^{\omega_n}x_n)$. ($Q=\sum_j\omega_j$ is called homogeneous dimension). Moreover, $(\tau_xf)(y)=f(x\circ y)$ is the (left) translation, and a differential operator P is left-invariant if $P(\tau f)=\tau(Pf)$. Assume that $\omega_1=\ldots=\omega_q$ for some $1<q<n$ and there are left-invariant differential operators $X_1,\dots X_q$ such that $X_j(0)=\partial_{x_j}$ and the Lie algebra generated by the X_j is $\mathbb{R}^n$ (Hörmander condition). $\mathbb{G}$ is of *step 2* if $\mathbb{R}^n=\mathrm{span}\,\{X_j,[X_i,X_j]\}$. Let $L=\sum_j X_j^2$ be the *(sub)laplacean*, ∂_t-L the *heat operator* and

$$W_tu(x)=\int_{\mathbb{R}^n}h(t,y^{-1}\circ x)u(y)dy \tag{5}$$

the heat semigroup. Here h is the *heat kernel* in $\mathbb{G}$. The main results in [8] are the following:

Theorem 1. *There is $c(\mathbb{G})\ge 0$ such that $|D_{\mathbb{G}}u|(\mathbb{R}^n)\le(1+c(\mathbb{G}))|D_{\mathbb{G}}u|(\mathbb{R}^n)$. Moreover, if $\mathbb{G}$ is step 2, setting $\pi(x_1,\dots,x_n)=(x_1,\dots,x_q)$ and*

$$\phi_{\mathbb{G}}(\nu)=\int_{T_{\mathbb{G}}(\nu)}h(1,x)dx,\quad \nu\in\mathbb{R}^q,\quad T_{\mathbb{G}}(\nu)=\{x\in\mathbb{R}^n:\ \langle\pi(x),\nu\rangle=0\}$$

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the following equality holds for every $u \in BV(\mathbb{G})$:

$$\int_{\mathbb{R}^n} \phi_{\mathbb{G}}(\sigma_u) d\,|D_{\mathbb{G}} u| = \lim_{t\to 0} \frac{1}{4\sqrt{t}} \int_{\mathbb{R}^n\times\mathbb{R}^n} |u(x)-u(y)| h(t, y^{-1}\circ x) dx dy. \tag{6}$$

Functions of bounded variation can be defined also in an infinite dimensional separable Banach space X endowed with a Gaussian measure $\gamma = \mathcal{N}(0,Q)$ (centred normal distribution with covariance Q) and a differential strucure coming from either the *Malliavin calculus*, see [11], [4], [5], [6] or a kind of Fréchet derivative if X is a Hilbert space, see [1]. These two points of view are related to different stochastic processes that are solutions of a particular stochastic differential equations and give rise to different related *Orstein-Uhlenbeck* semigroups. Problems analogous to (3) can be posed in this context. Let X be a Hilbert with inner product $\langle\cdot,\cdot\rangle$, norm $|\cdot|$, an orthonormal basis (e_k) such that $Qe_k = \lambda_k e_k$ for all $k \geq 1$, with λ_k a nonincreasing sequence of strictly positive numbers such that $\sum_k \lambda_k < \infty$. For $k \geq 1$, $f : X \to \mathbb{R}$, define the partial derivatives

$$D_k f(x) = \lim_{t\to 0} \frac{f(x+te_k) - f(x)}{t} \tag{7}$$

(provided that the limit exists) and, by linearity, the gradient operator $D : \mathcal{F}C_b^1(X) \to \mathcal{F}C_b(X,X)^2$. The gradient turns out to be a closable operator with respect to the topologies $L^p(X,\gamma)$ and $L^p(X,\gamma,X)$ for every $p \geq 1$. This leads to the integration by parts formula

$$\int_X \psi D_k \varphi d\gamma = -\int_X \varphi D_k^* \psi d\gamma, \quad \text{where } D_k^*\varphi = D_k\varphi - \frac{x_k}{\lambda_k}\varphi. \tag{8}$$

Accordingly, the *total variation* of a function $u \in L^2(X,\gamma)$ can be defined as

$$|D_\gamma u|(X) = \sup\Big\{\int_X u\Big[\sum_k D_k^*\phi_k\Big] d\gamma, \phi \in \mathcal{F}C_b^1(X,X), |\phi(x)| \leq 1\ \forall x \in X\Big\}. \tag{9}$$

As in the Euclidean case (but with a different proof) we know that if $D_\gamma u|(X) < \infty$ then there is a X-valued measure ν^u on X such that

$$\int_X u(x) D_k\varphi(x) d\gamma = -\int_X \varphi(x) d\nu_k^u + \frac{1}{\lambda_k}\int_X x_k u(x)\varphi(x) d\gamma, \qquad \varphi \in \mathcal{F}C_b^1(X),$$

with $\nu_k^u = \langle \nu^u, e_k\rangle_X$. In this case, introducing the semigroup

$$R_t f(x) = \int_X f(y) d\mathcal{N}(e^{tA}x, Q_t)(y) = \int_X f(e^{tA}x + y) d\mathcal{N}(0,Q_t)(y), \quad f \in B_b(X), \tag{10}$$

where

$$Q_t = \int_0^t e^{2sA} ds = -\frac{1}{2}\,A^{-1}(1 - e^{2tA}) \qquad \text{for} \quad A = -\frac{1}{2}Q^{-1},$$

$\mathcal{N}(0,Q_t) \xrightarrow{w^*} \mathcal{N}(0,Q) = \gamma$ as $t \to \infty$, so that γ is invariant for R_t. Then, $u \in BV_X(X,\gamma)$ if and only if

$$L(u) := \lim_{t\downarrow 0} \int_X |e^{-tA} D R_t u| d\gamma < \infty \tag{11}$$

and in this case $L(u) = |D_\gamma u|(X)$. The semigroup R_t comes from the stochastic differential equation $d\xi = AXdt + dW(t), \quad \xi(0) = x \in X$, where W is a cylindrical Brownian motion. More generally, in the paper [2] the case of a *log-concave* measure μ is considered. In this case the starting stochastic equation is $d\xi = (A\xi - DU(\xi))dt + dW(t), \quad \xi(0) = x$, where A and W are as above and the potential U belongs to $C^3(X)$, is convex, and D^2U, D^3U are uniformly continuous and bounded. Consider X endowed with the same differential structure and the probability measure $\mu(dx) = Z^{-1}e^{-2U(x)}\gamma(dx)$, where Z is the normalisation constant. Then, related to the above SDE is the semigroup $P_t\varphi(x) = \mathbb{E}[\varphi(X(t,x))]$, φ bounded and borel on X, and the $BV(X,\mu)$ space can be introduced through the variation

$$D_\mu u(X) = \sup\left\{\int_X \langle u(x), \mathrm{div}_\mu \phi(x)\rangle\, \mu(dx) : \in \mathcal{F}C_b^1(X;X),\ |\phi(x)| \leq 1\right\}. \tag{12}$$

where $D_k^*\varphi = -D_k\varphi - \varphi\, D_k \log\rho + \frac{1}{\lambda_k} x_k\,\varphi$ for $\rho = e^{-U}$. As before, $u \in BV(X,\mu)$ iff $D_\mu u(X) < \infty$.

[2] $\mathcal{F}C_b^k(X)$ is the space of the *cylindrical* real functions on X, i.e., $\phi \in \mathcal{F}C_b^k(X)$ if there are $m \in \mathbb{N}$, $x_1^*, \dots x_m^* \in X^*$ and $f \in \mathcal{F}C_b^k(\mathbb{R}^m)$ such that $\phi(x) = f(\langle x, x_1^*\rangle, \dots, \langle x, x_m^*\rangle)$.

Theorem 2. *A function $u \in L^2(X,\mu)$ belongs to $BV(X,\mu)$ if and only if there exists a vector measure ν^u such that*

$$\int_X u(x)\, D_k^*\varphi(x)\,\mu(dx) = \int_X \varphi(x)\nu_k^u(dx), \quad \forall\varphi \in C_b^1(X), \quad k \in \mathbb{N}. \tag{13}$$

In this case, the following equality holds:

$$\lim_{t\to 0}\int_X |DP_t u|d\mu = |D_\mu u|(X). \tag{14}$$

As a final remark, let us point out that the main result in [14], [9] has recently been generalised to a wider class of Riemannian manifolds by B. Güneysu and the author, and that the results briefly described here, in spite of their analogies, require in each case different (and in some cases very different) proofs.

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