Normal enveloping algebras

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Let A be an algebra with involution $*$ over a field $\mathbb{F}$. We recall that A is said to be normal if $xx^* = x^*x$ for every $x \in A$. Over the decades, normal algebras with involutions have been extensively investigated on their own (see e.g. [1], [3], [4], [5], [7], [8], [9], [10], [11]) and, moreover, they have several applications in linear algebra and functional analysis as well (see e.g. [2], [6], [11], [12], [13]). It is well-known that any normal algebra with involution satisfies the standard polynomial identity of degree 4 (see [7, §5]). Moreover, in [11] Maxwell determined the structure of a normal simple algebra of matrices with entries in a field with involution. He also proved that a division algebra D with involution is normal if and only if D is either a field or a generalized quaternion algebra over its center. Furthermore, a characterization of group algebras which are normal under the standard involution was established by Bovdi, Gudivok, and Semirot in [3]. Subsequently, such a result has been extended to twisted group algebras in [4] and to group algebras under a Novikov involution in [5].

On the other hand, it seems that the rather natural problems of characterizing ordinary and restricted enveloping algebras which are normal under their canonical involutions have not been settled yet. The present paper is just devoted to answering these questions.

For an arbitrary Lie algebra L we denote by $U(L)$ the universal enveloping algebra of L. Moreover, if L is restricted with a p-map $[p]$ over a field $\mathbb{F}$ of characteristic $p > 0$, then we denote by $u(L)$ the restricted enveloping algebra of L. We consider $U(L)$ and $u(L)$ with the *principal involution* $*$, namely, the unique $\mathbb{F}$-antiautomorphism such that $x^* = -x$ for every x in L. Note that $*$ is just the antipode of the $\mathbb{F}$-Hopf algebras $U(L)$ or $u(L)$.

We use the symbols $Z(L)$ and L' for the center and the derived subalgebra of L, respectively. If $S \subseteq L$ then we denote by $\langle S\rangle_{\mathbb{F}}$ and $\langle S\rangle$ the $\mathbb{F}$-vector space and the subalgebra generated by S. Also, if L is restricted then $\langle S\rangle_p$ denotes the restricted subalgebra generated by S and we put $S^{[p]} = \{x^{[p]} \,|\, x \in S\}$. In our first main result we completely settle the restricted case:

Theorem 1. *Let L be a restricted Lie algebra over a field $\mathbb{F}$ of characteristic $p > 0$. Then $u(L)$ is normal if and only if either L is abelian or $p = 2$, L is nilpotent of class 2 and, moreover, one of the following conditions holds:*

(i) *L contains an abelian restricted ideal I of codimension 1;*

(ii) $\dim_{\mathbb{F}} L/Z(L) = 3$*;*

(iii) $\dim_{\mathbb{F}} L' = 1$ *and* $(L')^{[2]} = 0$*;*

(iv) $L = \langle x, x_1, x_2, x_3\rangle_p + Z(L)$ *with* $[x_1, x_2] = \xi[x, x_3]$, $[x_1, x_3] = \mu[x, x_2]$, $[x_2, x_3] = \lambda[x, x_1]$, *and* $\lambda[x, x_1]^{[2]} + \mu[x, x_2]^{[2]} + \xi[x, x_3]^{[2]} = 0$ *for some* $\lambda, \mu, \xi \in \mathbb{F}$.

Afterwards, we apply Theorem 1 in order to solve the ordinary case:

Theorem 2. *Let L be a Lie algebra over an arbitrary field $\mathbb{F}$. Then $U(L)$ is normal if and only if either L is abelian or $p = 2$, L is nilpotent of class 2 and, moreover, one of the following conditions holds:*

(i) *L contains an abelian ideal of codimension 1;*

(ii) $\dim_{\mathbb{F}} L/Z(L) = 3$.

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