# Waves in the Skyrme–Faddeev model

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In classical nonlinear field equations certain classes of localized solutions are interpreted as particle-like excitations, when they are related to the existence of a topological index, which prevents the decaying into a superposition of elementary wave-like particles. For this reason such solutions are known as topological solitons. A great variety of field models admitting topological solitons have been studied (monopoles, skyrmions, instantons and vortices), playing an important role in High Energy Physics [1,2], General Relativity [3], as well as in Condensed Matter Physics [4]. A special interest deserves a 3D nonlinear sigma model, called *Skyrme - Faddeev model* [5], which it was proved [2] to be a special subcase of the pure quantum $SU\,(2)$-Yang-Mills theory in the infrared limit. Imposing suitable boundary conditions to the field configurations, the homotopy group of the theory results $\pi_3\left(S^2\right) = \mathbb{Z}$ and one can conclude that all solutions of the E.-L. equations are labelled by the integer Hopf index $Q$. It provides the linking number of the pre-images of two independent points on the target space $S^2$. Numerical calculations [6] have produced a comprehensive analysis of topological solitons with $1 \leq Q \leq 16$, proving the existence of local energy minima of knotted toroidal shape, possibly many times tangled ($Q = 7$ corresponds to a trifoil knot). Global analytical considerations [7] have shown the bounded from below by $S_{SF} \geq C\ \pi^2\ \rho\ |Q|^{3/4}$. Its main consequence is that vortices of higher topological charge are metastable configurations. Finally the characteristic size of a generic but stable perturbation is $\frac{1}{\rho} \leq R_{knot} \leq \frac{\sqrt{2}}{\rho}$. However, also space extended structures were considered in [8] and in [9]. Later [10] it was shown that collections of localized objects may condensate in order to form periodic structures in the space. Moreover , as pointed out in [8,10] the appearance of extended multi-sheeted structures may be energetically more favorable. Thus, the quest for periodic solutions (possibly exact) for the Skyrme–Faddeev model becomes more compelling. On the other hand, in a series of papers [11] it was shown that one may obtain completely integrable sub-systems, by adding certain differential constraints to the model. The integrability corresponds to the existence of infinitely many local conservation laws for the sub-system. Here we report the results contained in the article [12].

We analyze the 4-dimensional relativistic generalization of the Skyrme - Faddeev model in the space endowed with the pseudo-riemannian metric $\mathrm{diag}\,(g_i) = (+,-,-,-)$. The unimodular vector-field in polar representation $\vec{\phi} = (\sin w \cos u, \sin w \sin u, \cos w)$ is determined by the Lagrangian density

$$\mathcal{L}_p = \frac{1}{32\pi^2}\ \{w_\mu w^\mu + \sin^2 w\, [u_\mu u^\mu - \epsilon\,(w_\mu w^\mu u_\nu u^\nu - w_\mu w_\nu u^\mu u^\nu)]\}\,,$$

where $\epsilon > 0$ is the breaking-scale parameter of the model. The symmetry group is $\left(\mathbb{R}^4 \rtimes SO\,(3,1)\right) \otimes SO\,(3)$.

A first observation comes from the assumption $w = const$, thus the above system drastically reduces to the system given by the d'Alembert and the homogeneous Eikonal equations

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = 0,$$

the general solution of which is in the implicit form

$$G\,(u, A_\mu\,(u)\,x^\mu, B_\mu\,(u)\,x^\mu) = 0,$$
$$A_\mu A^\mu = B_\mu B^\mu = A_\mu B^\mu = 0,$$

with $G$, $A_\mu$ and $B_\mu$ arbitrary real regular functions [13]. The process to provide explicit form for $u$ may leads to multi-valued solutions, like as for shock waves. The corresponding differentiability singularity in the $\vec{\phi}$ field describes a type of a domain wall.

A more general reduction can be obtained by imposing

$$w_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha \quad (\alpha = \text{ constant } \in \mathbb{R})\,.$$

Then, the equations for the S-F model reduce to the equations

$$\partial_\mu u^\mu = 0, \qquad u_\nu u^\nu = \alpha, \quad w_\mu u^\mu = 0,$$

which are highly nonlinear for the $w$ field. Here we notice that the eikonal condition above were imposed by in [11], after different kinds of considerations. The general solution of the d'Alembert-Eikonal sub-system for $u$ is given in implicit form by [13]

$$\begin{aligned} u &= A_\mu(\tau)\, x^\mu + R_1(\tau), \\ & \quad B_\mu(\tau)\, x^\mu + R_2(\tau) = 0, \\ A_\mu A^\mu &= a, \quad A_\mu B^\mu = A'_\mu B^\mu = B_\mu B^\mu = 0, \end{aligned}$$

where the function $\tau$ is implicitly defined by the second relation and the arbitrary real regular functions $A_\mu$, $B_\mu$, $R_i$ satisfy the constraints in the third line.

Alternatively, one can set to zero the coefficients of all functions of $w$ in the Euler-Lagrange equations and select the most interesting reductions. Among them we will consider the reduced S-F system, which is the set of equations

$$\begin{gathered} \partial_\mu w^\mu = 0, w_\mu w^\mu = -\epsilon, u_\mu w^\mu = 0, \\ u_\nu \partial_\mu (w^\mu u^\nu - w^\nu u^\mu) = 0, \\ \epsilon \partial_\mu u^\mu + w_\nu \partial_\mu (u^\mu w^\nu - u^\nu w^\mu) = 0. \end{gathered}$$

The first two equations in are the d'Alembert-Eikonal system, whose general solution was shown above. The third one can be interpreted as an orthogonality condition among the gradients of the two fields. The last equation can be proved to just an identity, while the fourth one is a quadratic differential constraint among the derivatives of the function $u$, which can be written as

$$a_\mu w^\mu = 0 \text{ with } \quad a = u^\nu u_\nu,$$

Summarizing, the above overdetermined system describes completely the reduced Skyrme–Faddeev system. The compatibility conditions for the d'Alembert-Eikonal system subsystem is is well known Monge-Ampére equation $\mathrm{Det}\,[w_{ij}] = 0$. The compatibility conditions for $u$ are

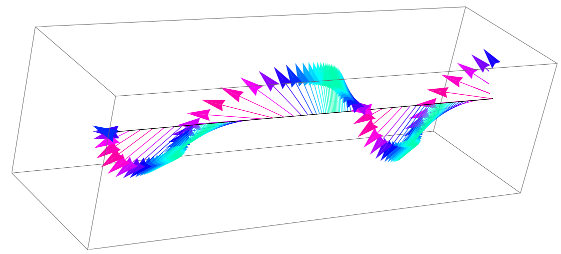
$$\begin{gathered} (w_s^2 - \epsilon) u_m u_k w_{km} + (u_k w_k)^2 w_{mm} = 2 u_s w_s u_m w_k w_{km}, \\ 4 u_k w_k u_s w_{sp} (w_m w_{pm} - w_p w_{mm}) + 2 (u_s w_m w_{sm})^2 + \\ (u_s w_s)^2 (w_{mm} w_{pp} - w_{pm}^2) = 2 (w_p^2 - \epsilon)(u_s w_{sm})^2. \end{gathered}$$

Then, a special class of solutions for $u$ is given by

$$u = F\,[w_1, w_2, w_3]\,.$$

Looking for the simplest generalization of the plane - wave solutions, one assumes that $w = \Theta\,[\theta]\,,\ u = \Phi\,[\theta] + \tilde{\theta}$, where $\theta = \alpha_\mu x^\mu$ (the *phase*) and $\tilde{\theta} = \beta_\mu x^\mu$ ( the *pseudo-phase*). Using the conservation laws, the equations of motion reduce to only one ODE in the independent phase $\theta$ for the auxiliary function $\psi = \sin\Theta$, given by

$$\psi_\theta^2 = \frac{64(\psi - 1)\,(\psi - A_1)\,(\psi - A_2)}{\text{[illegible]}},$$



where all other quantities are parameters related to the initial data. The general solution of the above equation is given in terms of incomplete elliptic integrals of third kind, leading to a large zoo of solitonic and periodic solutions. One of them is represented in the original $\vec{\phi}$ variable in Fig. 1. These solutions mimic the spin wave configurations called cyclon and extra-cyclon in multiferroic materials [14].

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