Analyticity of semigroups generated by a class of degenerate evolution equations on domains with corners

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The analyticity in the space of continuous functions on the d-dimensional canonical simplex S^d of the semigroup generated by the multi-dimensional Fleming-Viot operator (also known as Kimura operator or Wright-Fischer operator) has been a long-time open problem. These operator is defined by

$$Au(x) = \frac{1}{2}\sum_{i,j=1}^{d} x_i(\delta_{ij} - x_j)\partial^2_{x_i x_j} u(x) + \sum_{i=1}^{d} b_i(x)\partial_{x_i} u(x), \tag{1}$$

where $b = (b_1, \ldots, b_d)$ is a continuous inward pointing drift on the canonical simplex S^d in R^d. The operator (1) arises in the theory of Fleming–Viot processes as the generator of a Markov C_0–semigroup defined on $C(S_d)$. Fleming–Viot processes are measure–valued processes that can be viewed as diffusion approximations of empirical processes associated with some classes of discrete time Markov chains in population genetics. We refer to the book of Ethier-Kurtz quoted in [1] for more details on the topic. Nevertheless, it can be interesting to see how the peculiar geometry of the domain arises.

Assume that $A_1, \ldots, A_{d+1}$, $2 \le d < \infty$ are the various possibilities of one genetic character (e.g. the colour of eyes) in a population of individuals. An individual is called *of type* p if it has the character A_p. The assumptions about the population of individuals are that:

- generations do not overlap, each individual behaves indipendently of others, the size N of the population is conditioned to be fixed over generations.
- the offsprings are affected by
  - **selection**: a parent of type p produces only the same type of offspring individuals, whose number depends on a parameter $\xi_p > 0$ (fitness of type p). The larger is ξ_i w.r. to the other ξ_p, the more advantageous is the type i. In particular, if $\xi_1 = \ldots = \xi_d$, then there is neither selective advantage nor disadvantage among the alleles.
  - **mutating pressure**: an offspring of type p remains of type p with probability m_{pp} and it becomes of type q with probability m_{pq}. Clearly $m_{pq} \ge 0$, and $\sum_{q=1}^{d} m_{pq} = 1$. If $m_{pp} = 1$, the mutating pressure does not affect to type p.
  - **migration**: during each period of generation, there are individuals migrating from outside population into our population and their number is described by stochastic variable distributed according Poisson distribution with parameter $c_p \ge 0$. There is no immigrant of type p if $c_p = 0$.

If we denote by $X_p^{(N)}(k)$ the number of individuals of type p of the k-th generation and by

$$X^{(N)}(k) = (X_1^{(N)}(k)), \ldots, N - X_1^{(N)}(k) - \ldots - X_d^{(N)}(k)), \qquad k \in \mathbb{N}$$

then $\{X^{(N)}(k)\}_{k=0}^{\infty}$ is a Markov chain, with state space

$$\Omega_d^{(N)} = \{(\alpha_1, \ldots, \alpha_{d+1}) \mid \alpha_i \in \mathbb{N}, \sum_{i=1}^{d+1} \alpha_i = N\}$$

or, equivalently,

$$S_d^{(N)} = \{(\alpha_1, \ldots, \alpha_d) \mid x_i \in \mathbb{N}, \sum_{i=1}^{d} \alpha_i \le N\}$$

and set of times (the generations) $\mathbb{N}$, and is called multi-type frequency model of Wright-Fischer. Under some uniformity conditions, it is possible to perform a space-time rescaling of $X^{(N)}(k)$ and find reasonable continuous approximations of the above discrete time model.

More precisely we assume that

- $\sigma_p = N(\xi_p - 1)$ indipendent of N
- $\nu_{pq} = N(m_{pq} - \delta_{pq})$ indipendent of N
- c_p indipendent of N

Let

$$S_d := \{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid x_i \geq 0, \sum_{i=1}^{d} x_i \leq 1\}$$

and

$$\mathcal{A}u = \sum_{i,j=1}^{d} (\delta_{ij} x_i - x_i x_j) D_{ij} u + \sum_{i=1}^{d} \left(c_i - x_i \sum_{j=1}^{d+1} c_j + \sum_{j=1}^{d+1} (\delta_{ij} x_i - x_i x_j) \sigma_j + \sum_{j=1}^{d+1} \nu_{ij} x_j \right) D_i u$$

with $x_{d+1} = 1 - x_1 - \ldots - x_d$. Then the following results hold.

Theorem 1. *(Ethier-Kurtz, Shimakura, Fleming-Viot)*

1. *The closure of $(\mathcal{A}, C^2(S_d))$ generates a C_0-contraction semigroup $(T(t))_{t\geq 0}$ in $C(S_d)$.*
2. *Let $P^{(N)} = P^{(N)} = (P^{(N)}_{\alpha,\beta})_{\alpha,\beta \in S_d^{(N)}}$ be the one-step transition probability matrix of the Wright-Fischer model, i.e.*

   $$P^{(N)}_{\alpha,\beta} = P[X^{(N)}(k+1) = \beta \,|\, X^{(N)}(k) = \alpha]$$

   Set $[u] = (u\left(\frac{\alpha}{N}\right))_{\alpha \in S_d^{(N)}}$. For every $f \in C(S_{d-1})$, every $\varepsilon > 0$ and every $K \in \mathbb{N}$ there exists N_0 such that for every $N \geq N_0$ and very $k \leq KN$:

   $$|(P^{(N)})^k [f] - [T(k/N) f]| < \varepsilon \qquad \textit{for all } \alpha \in S_d^{(N)}$$

The semi-group $(T(t))_{t\geq 0}$ is called a *diffusion approximation* of the discrete model $P^{(N)}$ as N is large. The described construction thus motivates the study of the operators of type (1).

These operators have been largely studied using an analytic approach by several authors in different settings. The difficulty in the investigation of these operators is twofold: the operators (1) degenerate on the boundary of S_d in a very natural way and the domain S_d is not smooth as its boundary presents edges and corners. Thus the classical techniques for the study of uniformly elliptic operators on smooth domains cannot be applied.

In the one-dimensional case, the study of such type of degenerate (parabolic) elliptic problems on $C([0,1])$ started in the fifties with the papers by Feller and was clarified in the subsequent work of Clément and Timmermans. Also the problem of the regularity of the generated semigroup in $C([0,1])$ has been considered by several authors In particular, Metafune obtained the analyticity of the semigroup generated by $x(1-x)D^2$ on $C([0,1])$ in 1998, which was a problem left open for a long time.

In the d–dimensional case, Ethier proved the existence of a C_0–semigroup of positive contractions on $C(S_d)$ under mild conditions on the drift terms b_i, while Shimakura gave concrete representation formulas for the semigroups of diffusion processes associated to a class of Wright–Fisher models including the simplest case $b_i = 0$.

Few results about the regularity of the generated semigroup in $C(S_d)$ were known, until the paper [1], where we proved the analyticity of the semigroup generated by the operator in the case $b_i = 0$ The proof of the result is based on a sort of self-similarity of the operator on the faces of the simplex, that permits to proceed by induction on the integer d, on a "'freezing coefficients"' argument and on the proof of suitable gradient estimates.

The next point would be to extend this result to the case of a non-vanishing drift. To this end, it is needed a careful estimate of the constants appearing in the study of the sectoriality of the one-dimensional operator

$$\mathcal{A}u(x) = \gamma(x)x(1-x)u''(x) + b(x)u'(x), \qquad x \in [0,1], \tag{2}$$

where γ is a continuous strictly positive function and b is a continuous function such that $b(0) \geq 0$ and $b(1) \leq 0$. This study is performed in the paper [2].

Always keeping in mind that our target is to prove the analyticity in $C(S^d)$ of the semigroup generated by the operator (1), in [2] we deal with the same problem for the class of degenerate second order elliptic differential operators

$$\mathcal{L} = \Gamma(x)\sum_{i=1}^{d}[\gamma_i(x_i)x_i\partial^2_{x_i} + b_i(x)\partial_{x_i}], \quad x \in Q^d = [0,M]^d,$$

where $M > 0$, Γ, b_i and γ_i, for $i = 1,\ldots,d$, are continuous functions on Q^d and on $[0,M]$ respectively and $b = (b_1,\ldots,b_d)$ is an inward pointing drift.

These results will play an essential role, in a forthcoming paper, in dealing with the analogous problem on the simplex.

REFERENCES

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