Linear chaos

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It is commonly believed that chaos is linked to non-linearity and that it is self-evident that a linear system behaves in a predictable manner. However, in 1929, G.D. Birkhoff obtained an example of a linear operator that possesses an important ingredient of chaos: the existence of a dense orbit. In 1952 G.R. MacLane proved that the same phenomenon occurs for the fundamental operator in analysis: the differentiation operator. Further on S. Rolewicz (1969) showed that not only nonlinear shifts but also linear shifts can have dense orbits. As a consequence of these examples, researchers began in the nineteen-eighties to study the dynamical properties of general linear operators; henceforth, operators with a dense orbit were called hypercyclic. In 1991 G. Godefroy and J.H. Shapiro proposed to accept Devaneys definition of (nonlinear) chaos as the right definition for linear chaos: a linear operator is chaotic if it has a dense orbit, it has a dense set of periodic points and it has sensitive dependence on initial conditions. They then showed that many linear operators are chaotic, including the three classical operators of Birkhoff, MacLane and Rolewicz. The fact that chaos for linear systems has only been discovered recently is easily explained: as Rolewicz showed, hypercyclicity, and hence also linear chaos, requires an infinite-dimensional setting. In the last two decades linear dynamics has turned to be a remarkably active field of research that involves several area of mathematics including operator theory, complex analysis, ergodic theory and partial differential equations. We refer to the recent monographs [2,3].

Recently another notion of chaos has been studied in the infinite-dimensional linear setting, namely distributional chaos. This concept was introduced by Schweizer and Smítal in 1994 for interval maps with the aim of unifying various notions of chaos. Actually, it strengths the notion of Li-Yorke chaos. Distributional chaos for operators on Banach spaces has been characterized in a paper by Martinez-Gimenez, Oprocha and Peris. In connection with the previous properties, it is worth mentioning that there is no implication, in general, between hypercyclicity and distributional chaos and it is still open whether every Devaney chaotic operator on a Banach (or Hilbert) space is distributionally chaotic.

The main interest of the authors in this field is the continuous analogue of hypercyclic and chaotic operators in the form of semigroups. While the theories run parallel in great parts, hypercyclic and chaotic semigroups have important applications to partial differential equations.

These type semigroups were studied in a systematic way for the first time in the paper by Desch, Schappacher, and Webb in 1997, where they also give a sufficient condition for chaoticity of a semigroups based on the analysis of the point spectrum of the generator of the semigroup. Since then, it has been shown that chaos appears in C_0-semigroups associated to birth and death equations for cell populations, transport equations, first order partial differential equations and diffusion operators as the Ornstein-Uhlenbeck operators.

We recall that, if X is a Banach space, a one–parameter family $\mathcal{T} = \{T_t\colon X \to X \ ; \ t \geq 0\}$ is said to be a *strongly continuous semigroup of operators in* $L(X)$ (briefly, C_0*-semigroup of operators*) if the following conditions are satisfied.

(1) $T_0 = I$.

(2) $T_t T_s = T_{t+s}$, for all $s,\ t \geq 0$.

(3) $\lim_{t\to s} T_t x = T_s x$, for all $x \in X$ and $s \geq 0$.

These families of operators appear as solutions of abstract Cauchy problems arising from various types of evolution equations.

In [1] the authors were mainly interested in distributionally chaotic semigroups. If A is a Lebesgue measurable subset of the real line, then the *upper density* is defined as

$$\overline{\text{Dens}}(A) = \limsup_{t\to\infty} \frac{\mu(A \cap [0,t])}{t},$$

where μ denotes the Lebesgue measure.

A C_0-semigroup of operators $\mathcal{T} = \{T_t\}_{t\geq 0}$ on X is said to be *distributionally chaotic* if there exist an uncountable subset $S \subset X$ and $\delta > 0$ such that, for each pair of distinct points $x, y \in S$ and for every $\varepsilon > 0$, we have

$$\overline{\text{Dens}}(\{s \geq 0 \ ; \ ||T_s x - T_s y|| > \delta\}) = 1, \quad \text{and}$$

$$\overline{\text{Dens}}(\{s \geq 0 \ ; \ ||T_s x - T_s y|| < \varepsilon\}) = 1.$$

The set S is said to be a *distributionally δ–scrambled set* for $\mathcal{T}$ and the pair $\{x, y\}$ a *distributionally chaotic pair* for $\mathcal{T}$. If the scrambled set S is dense on X, then we say that $\mathcal{T}$ is *densely distributionally chaotic*.

The first part of [1] is devoted to the study of the interplay between the discrete and continuous notion of distributional chaos. In the second part, sufficient conditions for distributional chaos of semigroups, based on the analysis of the spectrum of the generator of the semigroup, have been presented and applied to concrete exmaples.

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