Non Maxwellian behaviour and quasi-stationary regimes near modal solutions of the Fermi–Pasta–Ulam β-system

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In recent decades, the study of Hamiltonian many–body systems has undergone great development. These systems are indeed ubiquitous in many different fields of science. In this context, one of the most remarkable examples is provided by the Fermi–Pasta–Ulam (FPU) system [1]. It has been intensively studied since its proposal (see [2] for a review), and still represents an invaluable model for studying nonlinear phenomena.

A recent series of papers [3–7] has been devoted to the stability properties of an interesting class of solutions admitted by FPU–β system with periodic boundary conditions: the *one–mode solutions* (OMS) [8–10]. These are exact solutions, usually referred to by means of the values of the mode number $n = \frac{N}{4},\ \frac{N}{3},\ \frac{N}{2},\ \frac{2}{3}N,\ \frac{3}{4}N$, where N is the number of the particles of the system. These modes are named equivalently (as in the present paper)$\pi/2$, $2\pi/3$, π, etc.

In [6], the existence of a new stability threshold $\epsilon_2(N)$, apart the well–known one of low energy $\epsilon_1(N)$, has been discovered for the $\pi/2$ mode. Indeed, on increase of the energy density ϵ, the system experiences an abrupt transition from the region of chaotic behaviour to a region where the nonlinear mode solution becomes again stable. Technically, we have used a global indicator introduced in [5,6], i.e. the ratio ρ between the standard deviation and the first moment of the absolute value of the relevant variable for a given probability distribution. For a Gaussian distribution, $\rho = \sqrt{\pi/2}$. This indicator estimates the deviation of a generic assigned distribution from the Gaussian behaviour for any value of the excitation energy density. It is a function of the dynamical variables of the configuration space only and its usefulness relies on the fact that is model–independent. By means of this indicator in [5,6] we studied the stability of the π and $\pi/2$–modes as functions of the energy density.

The aim of our research is to explore the statistical properties of the FPU–β chain, following the orbits of the $\pi/2$ one–mode solutions.

In the last years, a new theoretical framework, called nonextensive statistical mechanics, has appeared for describing the thermostatistics of systems typically exhibiting long–range correlations, (asymptotic) scale invariance, multifractality, etc. [11,12]. The nonextensive scenario generalizes the classical Boltzmann–Gibbs (BG) statistics in the sense that it applies to non ergodic, e.g. weakly chaotic, systems (for a regularly updated bibliography, see [13]). The entropy on which it is based reads

$$S_q = K\frac{1-\sum_{i=1}^{W} p_i^q}{1-q}$$

$$(S_1 \equiv S_{BG} = -K\sum_{i=1}^{W} p_i \ln p_i)$$

where W is the total number of microscopic states of the system. This entropy, under suitable constraints, is extremized by a q–Gaussian distribution $\propto e_q(-\beta_q x^2) = \left[1-(1-q)\beta_q x^2\right]^{1/(1-q)}$, with $\beta_q > 0$. S_q is nonadditive, but for special values of the parameter q can be extensive, according to the prescription of Clausius [12]. A link between generalized entropies and number theory has been found [14]. In particular, the entropy S_q has been related to the classical Riemann zeta function.

A connection between the weakly chaotic dynamics of the model and nonextensive statistical mechanics was first established in [5] in the specific case of the π mode, for initial conditions in a narrow region of the phase space. This result has been also confirmed, for the same modal solution, by a subsequent analysis [15].

Recently [16] [M. Leo, R. A. Leo, P. Tempesta and C. Tsallis, Phys. Rev. E **85**, 031149 (2012] we have shown that striking evidence exists of the existence, for the $\pi/2$ mode, of *quasi–stationary states* whose thermostatistics is governed by a long standing q–Gaussian distribution for a considerable range of the energy density. This is achieved by performing an accurate analysis of a set of suitable observables associated with the evolution of the system. More precisely, for values of the energy density between ϵ_1 and ϵ_2, up to values very close to ϵ_2, the numerical distribution is fitted with a high accuracy by a q–Gaussian distribution, for values of N approximately up to 100, and very large integration times. Interestingly enough, the distributions found in our analysis are extremely stable, i.e, they remain q–Gaussian without converting into a Gaussian, or any different one. For N very large, q approaches 1, hence the q–Gaussian distribution essentially recovers the normal one.

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