Telescopic Creation for Identities of Rogers-Ramanujan Type †

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Summary The telescopic approach is proposed for creating new identities of Rogers-Ramanujan type from the known ones. Twenty-five examples are illustrated with several new identities being presented.

For two indeterminate x and q, the shifted factorial of x with base q is defined by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x) \quad \text{for} \quad n=1,2,\cdots.$$

When $|q|<1$, we have the following two well-defined infinite products

$$(x;q)_\infty = \prod_{k=0}^{\infty}(1-q^kx) \quad \text{and} \quad (x;q)_n = (x;q)_\infty \,/\, (q^nx;q)_\infty$$

where the multiparameter form for the former will be abbreviated to

$$(\alpha,\beta,\cdots,\gamma;q)_\infty = (\alpha;q)_\infty(\beta;q)_\infty\cdots(\gamma;q)_\infty.$$

Then the celebrated Rogers-Ramanujan identities [19, 21] may be reproduced as

$$\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q,q^4;q^5)_\infty} \quad \text{and} \quad \sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2,q^3;q^5)_\infty}$$

which have played important role in algebraic characters [15], partition theory [3, Chapter 7] and statistical mechanics [9]. There are numerous identities expressing infinite sums as infinite products, usually called identities of Rogers-Ramanujan type (shortly as RR-identities). For a comprehensive investigation, refer to Slater [24] in her collection of 130 identities and Sills [22], who has provided an excellent up-to-date annotation. Most recently, there is a dynamic survey by McLaughlin-Sills-Zimmer [17] and a systematic treatment through the bilateral Bailey lemma by Chu and Zhang [11], where two hundreds identities of Rogers-Ramanujan type have been tabulated.

The purpose of this paper is to present a telescopic approach to the identities of Rogers-Ramanujan type, which can briefly be described as follows. First write a genetic identity of Rogers-Ramanujan type as

$$\mathrm{PP} = \sum_{n=0}^{\infty}\mathrm{RR}_n. \tag{1}$$

Then for any sequence $\{T_n\}_{n\geq 0}$ convergent to zero, it is almost trivial to show, by means of the telescoping method, that there holds

$$0 \;=\; T_0 + \sum_{n=0}^{\infty}\Delta T_n \tag{2}$$

where $\Delta T_n := T_{n+1}-T_n$ stands for the usual forward difference. Finally adding the last two equations together leads us to a third one

$$\mathrm{PP} = T_0 + \sum_{n=0}^{\infty}\big\{\mathrm{RR}_n + \Delta T_n\big\}. \tag{3}$$

When the sequence $\{T_n\}_{n\geq 0}$ is appropriately devised, the last equation will often result in another RR-identity. We call this procedure *telescopic creation* for identities of Rogers-Ramanujan type.

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This procedure can be illustrated by the following example. In one of the pioneering papers of Bailey lemma and application, Bailey [8] stated several RR-identities. One of them (Equation 7.4) reads as

$$\begin{aligned}(1-q)\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty &= \sum_{n=0}^{\infty}\frac{(q;q^3)_{n+2}(q^2;q^3)_{n+1}}{(q^3;q^3)_n(q^3;q^6)_{n+2}}q^{3\binom{n+1}{2}} \qquad (4)\\ &= \sum_{n=0}^{\infty}(1-q^{3n})\frac{(q;q^3)_{n+1}(q^2;q^3)_n}{(q^3;q^3)_n(q^3;q^6)_{n+1}}q^{3\binom{n}{2}}\end{aligned}$$

where we have made the replacement $n\to n-1$ on the summation index. Now for the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=0 \quad\text{and}\quad T_n:=\frac{(q;q^3)_n(q^2;q^3)_n}{(q^3;q^3)_{n-1}(q^3;q^6)_n}q^{3\binom{n}{2}}$$

it is not hard to check that

$$\Delta T_n = q^{3\binom{n}{2}}\frac{(q;q^3)_{n+1}(q^2;q^3)_n}{(q^3;q^3)_{n-1}(q^3;q^6)_{n+1}}\Big\{\frac{q^{3n}(1-q)(1-q^{3n+2})}{(1-q^{3n})(1-q^{3n+1})}-1\Big\}.$$

Then the equation corresponding to (3) results in the identity [11, No 50]:

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty=\sum_{n=0}^{\infty}\frac{(q;q^3)_n(q^2;q^3)_{n+1}}{(q^3;q^3)_n(q^3;q^6)_{n+1}}q^{3\binom{n+1}{2}}. \qquad (5)$$

Furthermore, if we define another sequence $\{T'_n\}_{n\geq 0}$ by

$$T'_0=1 \quad\text{and}\quad T'_n:=\frac{(q;q^3)_n(q^2;q^3)_n}{(q^3;q^3)_n(q^3;q^6)_n}q^{3\binom{n+1}{2}}$$

then the corresponding forward difference reads as

$$\Delta T'_n = q^{3\binom{n+1}{2}}\frac{(q;q^3)_n(q^2;q^3)_{n+1}}{(q^3;q^3)_n(q^3;q^6)_{n+1}}\Big\{q^{3n+3}\frac{(1-q^{-1})(1-q^{3n+1})}{(1-q^{3n+2})(1-q^{3n+3})}-1\Big\}.$$

Applying (3) to (5) leads us to the following equation:

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty=1+\sum_{n=0}^{\infty}\frac{(q;q^3)_{n+1}(q^{-1};q^3)_{n+1}}{(q^3;q^3)_{n+1}(q^3;q^6)_{n+1}}q^{3\binom{n+2}{2}}.$$

Replacing n by $n-1$ yields another identity:

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}(q^2;q^2)_\infty=\sum_{n=0}^{\infty}\frac{(q^{-1};q^3)_n(q;q^3)_n}{(q^3;q^3)_n(q^3;q^6)_n}q^{3\binom{n+1}{2}}. \qquad (6)$$

This example suggests that the telescopic creation may potentially be useful for proving and discovering RR-identities, which will be confirmed by the rest of the paper. Following the same scheme, we shall present twenty-five further examples of other identities of Rogers-Ramanujan type. To our knowledge, most of the displayed RR-identities do not seem to have appeared previously except for those specified explicitly with references.

§1. **RR-identities related to modulo 2.** We start from the identity due to Slater [24, Eq 66]

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{\begin{matrix}\left(q^{16},q^6,q^{10};q^{16}\right)_\infty\\ +q\left(q^{16},q^2,q^{14};q^{16}\right)_\infty\end{matrix}\right\}=\sum_{n=0}^{\infty}\frac{(-1;q^4)_n\,q^{n^2}}{(-q^2;q^2)_n(q;q)_{2n}} \qquad (7)$$

which is equivalent to the following one by Chu-Zhang [11, No 11]

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(-q^4,-q,\,q^3;-q^4\right)_\infty=\sum_{n=0}^{\infty}\frac{(-1;q^4)_n\,q^{n^2}}{(-q^2;q^2)_n(q;q)_{2n}}. \qquad (8)$$

In view of Jacobi's triple product identity, this can be justified by the equation

$$\left(-q^4,-q,\,q^3;-q^4\right)_\infty=\left(q^{16},q^6,q^{10};q^{16}\right)_\infty+q\left(q^{16},q^2,q^{14};q^{16}\right)_\infty.$$

Similar identities with more factors can be found in Cooper and Hirschhorn [12].

Now define the sequence $\{T_n\}_{n\geq 0}$ by

$$T_0 = 0 \quad \text{and} \quad T_n := \frac{(-q^4;q^4)_{n-1}q^{n^2}}{(-q^2;q^2)_{n-1}(q;q)_{2n-1}}$$

and then compute the forward difference

$$\Delta T_n = \frac{(-q^4;q^4)_{n-1}\,q^{n^2}}{(-q^2;q^2)_n(q;q)_{2n+1}}\Big\{(1+q^{4n}) - 2(1-q^{2n+1})\Big\}.$$

In view of (3), we recover the following identity due to Chu-Zhang [11, No 13]:

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big(-q^4,-q,\,q^3;-q^4\big)_\infty = \sum_{n=0}^{\infty}\frac{(-q^4;q^4)_n\,q^{n^2}}{(-q^2;q^2)_n(q;q)_{2n+1}}. \tag{9}$$

§2. **RR-identities related to modulo 3.** Recall the identity number 6 in Slater [24] (see [6, Entry 4.2.8] and [18, Eq 29] also):

$$\frac{\big(q^3,-q,-q^2;q^3\big)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{(-1;q)_n\,q^{n^2}}{(q;q)_n(q;q^2)_n}. \tag{10}$$

Define the two sequences $\{T_n\}_{n\geq 0}$ and $\{T'_n\}_{n\geq 0}$ by

$$\begin{aligned} T_0 = 0 \quad &\text{and} \quad T_n := \frac{(-q;q)_{n-1}\,q^{n^2}}{(q;q)_{n-1}(q;q^2)_n},\\ T'_0 = 0 \quad &\text{and} \quad T'_n := \frac{(-1;q)_n q^{n^2}}{(q;q)_{n-1}(q;q^2)_n}; \end{aligned}$$

and then evaluate their differences

$$\begin{aligned} \Delta T_n &= \frac{(-1;q)_n\,q^{n^2}}{(q;q)_n(q;q^2)_n}\Big\{\frac{1+q^n}{2(1-q^{2n+1})} - 1\Big\},\\ \Delta T'_n &= \frac{(-1;q)_n\,q^{n^2}}{(q;q)_n(q;q^2)_n}\Big\{\frac{q^n(1+q^{n+1})}{1-q^{2n+1}} - 1\Big\}. \end{aligned}$$

According to (3), we recover respectively the following RR-identities

$$\frac{\big(q^3,-q,-q^2;q^3\big)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{(-q;q)_n\,q^{n^2}}{(q;q)_n(q;q^2)_{n+1}}, \tag{11}$$

$$\frac{\big(q^3,-q,-q^2;q^3\big)_\infty}{(q;q)_\infty} = \sum_{n=0}^{\infty}\frac{(-1;q)_n(-q;q)_{n+1}}{(q;q)_{2n+1}}q^{n(n+1)}; \tag{12}$$

where the first one can be found in Slater [24, Eq 26].

§3. **RR-identities related to modulo 4.** Recall the identity displayed in Chu-Zhang [11, No 19]:

$$2\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\big(q^4,\,-q^3,\,-q^5;q^4\big)_\infty = \sum_{n=0}^{\infty}\frac{(q^2;q^4)_n q^{n(n-2)}}{(q^4;q^4)_n(-q^{-1};q^2)_n}. \tag{13}$$

Define the sequence $\{T_n\}_{n\geq 0}$ by

$$T_0 = 0 \quad \text{and} \quad T_n := \frac{(q;q^2)_n(-q;q^2)_{n-1}q^{n(n-2)}}{(q^4;q^4)_{n-1}(-q^{-1};q^2)_n}$$

and then compute the forward difference

$$\Delta T_n = \frac{(q^2;q^4)_n q^{n(n-2)}}{(q^4;q^4)_n(-q^{-1};q^2)_n}\Big\{\frac{2q^{2n-1}}{1+q^{2n-1}} - 1\Big\}.$$

In view of (3), we obtain the following interesting identity:

$$\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\big(q^4,\ -q,\ -q^3;q^4\big)_\infty=\sum_{n=0}^{\infty}\frac{(q;q^2)_n}{(q^4;q^4)_n}q^{n^2} \tag{14}$$

which is equivalent to Stanton [25, P 61] (cf. Chu-Zhang [11, No 18] also).

§4. **RR-identities related to modulo 5: A.** From the following identity due to Rogers [20] (cf. [7, Eq 6.3], [14] and [24, Eq 15] also)

$$\frac{\big(q^5,\ q,\ q^4;q^5\big)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2-2n}}{(-q;q^2)_n(q^4;q^4)_n} \tag{15}$$

we can derive from (3) an identity due to Bailey [7, Eq 5.3] (see [13, Eq 7.11] and [18, Eq 62] also):

$$\frac{\big(q^5,\ q,\ q^4;q^5\big)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+2n}}{(-q;q^2)_{n+1}(q^4;q^4)_n}. \tag{16}$$

This is justified by defining the sequence $\{T_n\}_{n\geq 0}$ through

$$T_0=0 \quad\text{and}\quad T_n:=\frac{(-1)^n q^{3n^2-2n}}{(-q;q^2)_n(q^4;q^4)_{n-1}}$$

and then computing the forward difference

$$\Delta T_n=\frac{(-1)^n q^{3n^2-2n}}{(-q;q^2)_n(q^4;q^4)_n}\Big\{\frac{q^{4n}}{1+q^{2n+1}}-1\Big\}.$$

§5. **RR-identities related to modulo 5: B.** Similarly, combining an identity of Bailey [8, Eq 6.7] (cf. [13, Eq 7.7], [24, Eq 20] and [25, P 60] also)

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big(q^5,\ q^2,\ q^3;q^5\big)_\infty=\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q^4;q^4)_n} \tag{17}$$

with the sequence $\{T_n\}_{n\geq 0}$ given by

$$T_0=0 \quad\text{and}\quad T_n:=\frac{q^{n^2}}{(q^4;q^4)_{n-1}}$$

and the forward difference

$$\Delta T_n=\frac{q^{n^2}}{(q^4;q^4)_n}\Big\{(q^{2n+1}+q^{4n})-1\Big\}$$

we derive from (3) the following identity due to Chu-Zhang [11, No 30]:

$$\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\big(q^5,\ q^2,\ q^3;q^5\big)_\infty=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\ q^{n^2+2n}}{(-q^{-1};q^2)_n(q^4;q^4)_n}. \tag{18}$$

§6. **RR-identities related to modulo 6: A.** Recall an identity due to Slater [24, Eq 27] (see [18, Eq 32] also)

$$\frac{\big(q^6,-q,-q^5;q^6\big)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(-q;q^2)_{n+1}}{(q^2;q^2)_{2n+1}}q^{2n(n+1)}. \tag{19}$$

According to (3), we derive the following two new RR-identities

$$\frac{\big(q^6,-q,-q^5;q^6\big)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(-q^{-1};q^2)_n(-q^3;q^2)_n}{(q^2;q^2)_{2n}}q^{2n^2}, \tag{20}$$

$$2\frac{\big(q^6,-q,-q^5;q^6\big)_\infty}{(q^2;q^2)_\infty}=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q^2)_n(q^4;q^4)_n}q^{2n(n-1)}. \tag{21}$$

They are accomplished by defining the two sequences $\{T_n\}_{n\ge0}$ and $\{T'_n\}_{n\ge0}$ respectively through

$$T_0 = 1 \quad\text{and}\quad T_n := \frac{(-q;q^2)_n\, q^{2n(n+1)}}{(q;q^2)_n(q^4;q^4)_n},$$
$$T'_0 = 0 \quad\text{and}\quad T_n := -\frac{(-q;q^2)_n q^{2n(n-1)}}{2(q;q^2)_n(q^4;q^4)_{n-1}};$$

and then evaluating the following differences:

$$\Delta T_n = \frac{(-q;q^2)_n\, q^{2n(n+1)}}{(q;q^2)_{n+1}(q^4;q^4)_n}\Big\{\frac{q^{2n+1}(1+q^{2n+3})}{1-q^{4n+4}} - 1\Big\},$$
$$\Delta T'_n = \frac{(-q;q^2)_n q^{2n(n-1)}}{(q;q^2)_{n+1}(q^4;q^4)_n}\Big\{\frac{1-q^{2n+1}}{2} - q^{4n}\Big\}.$$

§7. **RR-identities related to modulo 6: B.** In accordance with the following identity due to Bailey [8, Eq 7.3]

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(q^6,\ q,\ q^5;q^6\right)_\infty = \sum_{n=0}^{\infty} \frac{(q;q)_{3n+1}(-q^3;q^3)_n}{(q^3;q^3)_{2n+1}(q^3;q^3)_n} q^{3\binom{n+1}{2}} \tag{22}$$

letting the sequence $\{T_n\}_{n\ge0}$ be defined by

$$T_0 = 1 \quad\text{and}\quad T_n := \frac{(q;q)_{3n}(-q^3;q^3)_n}{(q^3;q^3)_{2n}(q^3;q^3)_n} q^{3\binom{n+1}{2}}$$

and then computing the difference

$$\Delta T_n = \frac{(q;q)_{3n+1}(-q^3;q^3)_n}{(q^3;q^3)_{2n+1}(q^3;q^3)_n} q^{3\binom{n+1}{2}} \Big\{q^{3n+3}\frac{(1-q^{-2})(1-q^{3n+2})}{(1-q^{3n+3})(1-q^{3n+1})} - 1\Big\}$$

we find via (3) the following interesting RR-identity:

$$\frac{(-q^3;q^3)_\infty}{(q^3;q^3)_\infty}\left(q^6,\ q,\ q^5;q^6\right)_\infty = \sum_{n=0}^{\infty} \frac{(q^2;q^3)_n(q^{-2};q^3)_n}{(q^3;q^3)_n(q^3;q^6)_n} q^{3\binom{n+1}{2}}. \tag{23}$$

§8. **RR-identities related to modulo 8: A.** In view of the following identity due to Ramanujan [6, Entry 1.7.4]

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^4,\ q^4;q^8\right)_\infty = \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{\binom{n+1}{2}}}{(q;q)_n(q;q^2)_n} \tag{24}$$

define the sequence $\{T_n\}_{n\ge0}$ accordingly by

$$T_0 = 0 \quad\text{and}\quad T_n := \frac{(-1;q^2)_n q^{\binom{n+1}{2}}}{(q;q)_{n-1}(q;q^2)_n}$$

and check the forward difference

$$\Delta T_n = \frac{(-1;q^2)_n q^{\binom{n+1}{2}}}{(q;q)_n(q;q^2)_n}\Big\{\frac{q^n(1+q)}{1-q^{2n+1}} - 1\Big\}.$$

We therefore derive from (3) the following interesting identity:

$$\frac{(-q^2;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^4,\ q^4;q^8\right)_\infty = \sum_{n=0}^{\infty} \frac{(-1;q^2)_n q^{\binom{n}{2}+2n}}{(q;q)_n(q;q^2)_{n+1}}. \tag{25}$$

§9. **RR-identities related to modulo 8: B.** Applying (3) to another identity due to Ramanujan [6, Entry 1.7.5] (see [22, A105a] also)

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^2,\ q^6;q^8\right)_\infty = \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n\, q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}(q;q)_n} \tag{26}$$

results in the following RR-identity

$$\frac{(-q^2;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^2,\ q^6;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(-q;q)_n(-1;q^2)_n\,q^{\binom{n}{2}+2n}}{(-1;q)_n(q;q^2)_{n+1}(q;q)_n} \tag{27}$$

where we have defined the sequence $\{T_n\}_{n\geq 0}$ by

$$T_0=0\quad\text{and}\quad T_n:=\frac{(-q^2;q^2)_{n-1}\,q^{\binom{n+1}{2}}}{(q;q^2)_n(q;q)_{n-1}}$$

and evaluated its difference as follows:

$$\Delta T_n=\frac{(-q^2;q^2)_n\,q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}(q;q)_n}\Big\{\frac{q^n(1+q)(1+q^n)}{1+q^{2n}}-1\Big\}.$$

§10. **RR-identities related to modulo 8: C.** Recall an identity due to Slater [24, Eqs 37 and 105]

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^3,\ q^5;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n\,q^{\binom{n+1}{2}}}{(q;q^2)_{n+1}(q;q)_n}. \tag{28}$$

For the two sequences $\{T_n\}_{n\geq 0}$ and $\{T'_n\}_{n\geq 0}$ respectively given by

$$T_0=0\quad\text{and}\quad T_n:=-\frac{(-q;q^2)_n q^{\binom{n}{2}}}{2(q;q)_{n-1}(q;q^2)_n},$$

$$T'_0=1\quad\text{and}\quad T'_n:=\frac{(-q;q^2)_n q^{\binom{n+1}{2}}}{(q;q^2)_n(q;q)_n};$$

their differences read as follows:

$$\Delta T_n = \frac{(-q;q^2)_n\,q^{\binom{n+1}{2}}}{(q;q)_n(q;q^2)_{n+1}}\Big\{\frac{1-q^{2n+1}}{2q^n}-1\Big\},$$

$$\Delta T'_n = \frac{(-q;q^2)_n q^{\binom{1+n}{2}}}{(q;q)_n(q;q^2)_{n+1}}\Big\{\frac{q^{n+1}(1+q^n)}{1-q^{n+1}}-1\Big\}.$$

Then the corresponding (3) leads to the two RR-identities [11, No 65 and No 66]:

$$2\frac{(-q;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^3,\ q^5;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n}{(q;q)_n(q;q^2)_n}q^{\binom{n}{2}}, \tag{29}$$

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(q^8,\ q^3,\ q^5;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(-1;q)_n(-q^{-1};q^2)_n}{(-q^{-1};q)_n(q;q)_n(q;q^2)_n}q^{\binom{n+1}{2}}. \tag{30}$$

§11. **RR-identities related to modulo 8: D.** Rewrite the identity displayed in Chu-Zhang [11, No 80]

$$\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\left(q^8,\ -q^3,\ -q^5;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(1+q^{4n+3})(-q;q)_{4n}q^{2n(n+1)}}{(-q^2;q^4)_n(q^4;q^4)_{1+2n}}. \tag{31}$$

For the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=1\quad\text{and}\quad T_n:=\frac{(-q;q)_{4n}q^{2n(n+1)}}{(-q^2;q^4)_n(q^4;q^4)_{2n}}$$

the corresponding forward difference reads as

$$\Delta T_n=-\frac{(1+q^{4n+3})(-q;q)_{4n}q^{2n(n+1)}}{(-q^2;q^4)_n(q^4;q^4)_{2n+1}}\Big\{1-q^{4n+3}\frac{(1+q)(1+q^{4n+1})(1+q^{4n+4})}{(1+q^{4n+3})(1-q^{8n+8})}\Big\}.$$

Then applying (3) and replacing the summation index n by $n-1$, we find the following companion identity:

$$\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\left(q^8,-q^3,-q^5;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(-q^{-1};q^2)_{2n}q^{2n(n+1)}}{(q^4;q^4)_n(q^4;q^8)_n}. \tag{32}$$

§12. **RR-identities related to modulo 8: E.** In 1983, Gessel and Stanton [13, Eq 7.25] discovered the following strange identity

$$\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\left(q^8,\ q,\ q^7;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(q;q^2)_{2n+1}\,q^{2n(n+1)}}{(q^4;q^8)_{n+1}(q^4;q^4)_n}. \tag{33}$$

Letting the sequence $\{T_n\}_{n\geq 0}$ be defined by

$$T_0=1 \quad\text{and}\quad T_n:=\frac{(q;q^2)_{2n}\,q^{2n(n+1)}}{(q^4;q^8)_n(q^4;q^4)_n}$$

and then evaluating its difference by

$$\Delta T_n=\frac{(q;q^2)_{2n+1}q^{2n(n+1)}}{(q^4;q^8)_{n+1}(q^4;q^4)_n}\Big\{q^{4n+4}\frac{(1-q^{-3})(1-q^{4n+3})}{(1-q^{4n+1})(1-q^{4n+4})}-1\Big\}$$

we find through the creation procedure the interesting companion identity:

$$\frac{(-q^4;q^4)_\infty}{(q^4;q^4)_\infty}\left(q^8,q,q^7;q^8\right)_\infty=\sum_{n=0}^{\infty}\frac{(q^{-3};q^4)_n(q^3;q^4)_n}{(q^4;q^4)_n(q^4;q^8)_n}q^{2n(n+1)}. \tag{34}$$

§13. **RR-identities related to modulo 9: A.** Combining the following identity due to Bailey [7, Eq 1.7] (cf. [2, Eq 1.6] and [24, Eq 40] also)

$$\frac{\left(q^9,\ q,\ q^8;q^9\right)_\infty}{(q^3;q^3)_\infty}=\sum_{n=0}^{\infty}\frac{(q;q^3)_{n+1}(q^2;q^3)_n}{(q^3;q^3)_{2n+1}}q^{3n(n+1)} \tag{35}$$

with the sequence $\{T_n\}_{n\geq 0}$ given by

$$T_0=1 \quad\text{and}\quad T_n:=\frac{(q;q)_{3n}q^{3n(n+1)}}{(q^3;q^3)_{2n}(q^3;q^3)_n}$$

as well as the forward difference

$$\Delta T_n=-\frac{(q;q)_{3n+1}q^{3n(n+1)}}{(q^3;q^3)_{2n+1}(q^3;q^3)_n}\Big\{1+q^{3n+1}\frac{(1-q^{3n+2})(1-q^{3n+5})}{(1-q^{3n+1})(1-q^{6n+6})}\Big\}$$

we get the following interesting RR-identity:

$$\frac{\left(q^9,\ q,\ q^8;q^9\right)_\infty}{(q^3;q^3)_\infty}=\sum_{n=0}^{\infty}\frac{(q^{-2};q^3)_n(q^5;q^3)_n}{(q^3;q^3)_{2n}}q^{3n^2}. \tag{36}$$

§14. **RR-identities related to modulo 9: B.** Similarly, the combination of another identity due to Bailey [7, Eq 1.8] (see [2, Eq 1.5] and [24, Eq 41] also)

$$\frac{\left(q^9,\ q^2,\ q^7;q^9\right)_\infty}{(q^3;q^3)_\infty}=\sum_{n=0}^{\infty}\frac{(q;q^3)_n(q^2;q^3)_{n+1}}{(q^3;q^3)_{2n+1}}q^{3n(n+1)} \tag{37}$$

with the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=1 \quad\text{and}\quad T_n:=\frac{(q;q^3)_n(q^2;q^3)_n}{(q^3;q^3)_{2n}}q^{3n(n+1)}$$

and its forward difference

$$\Delta T_n=-q^{3n(n+1)}\frac{(q;q^3)_n(q^2;q^3)_{n+1}}{(q^3;q^3)_{2n+1}}\Big\{1+q^{3n+2}\frac{(1-q^{3n+1})(1-q^{3n+4})}{(1-q^{6n+6})(1-q^{3n+2})}\Big\}$$

yields another RR-identity of the same structure

$$\frac{\left(q^9,\ q^2,\ q^7;q^9\right)_\infty}{(q^3;q^3)_\infty}=\sum_{n=0}^{\infty}\frac{(q^{-1};q^3)_n(q^4;q^3)_n}{(q^3;q^3)_{2n}}q^{3n^2}. \tag{38}$$

§15. **RR-identities related to modulo 10.** According to (3), the identity due to Bailey [7, Eq 10.4] (see [1, Eq 2.2], [5, Eq 1.3], [13, Eq 7.18] and [24, Eq 46] also)

$$\frac{\left(q^{10},\ q^4,\ q^6;q^{10}\right)_\infty}{(q;q)_\infty}=\sum_{n=0}^{\infty}\frac{q^{n^2+\binom{n}{2}}}{(q;q)_n(q;q^2)_n} \tag{39}$$

is equivalent to the following one appeared in [4] and [1, Eq 2.3]:

$$\frac{\left(q^{10},\ q^4,\ q^6;q^{10}\right)_\infty}{(q;q)_\infty}=\sum_{n=0}^{\infty}\frac{q^{n^2+\binom{n+1}{2}}}{(q;q)_n(q;q^2)_{n+1}}. \tag{40}$$

This can be verified by defining the sequence $\{T_n\}_{n\geq 0}$ by

$$T_0=0 \quad\text{and}\quad T_n:=\frac{q^{n^2+\binom{n}{2}}}{(q;q)_{n-1}(q;q^2)_n}$$

and then evaluating its difference as follows:

$$\Delta T_n=\frac{q^{n^2+\binom{n}{2}}}{(q;q)_n(q;q^2)_n}\Big\{\frac{q^n}{1-q^{2n+1}}-1\Big\}.$$

§16. **RR-identities related to modulo 12: A.** According to an identity due to Slater [24, Eqs 55 and 57] (see [10, Eq 1.22b] and [18, Eq 37] also)

$$\frac{\left(q^{12},\ q,\ q^{11};q^{12}\right)_\infty}{(q^4;q^4)_\infty}=\sum_{n=0}^{\infty}\frac{(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}}q^{4n(n+1)} \tag{41}$$

we may define the sequence $\{T_n\}_{n\geq 0}$ by

$$T_0=1 \quad\text{and}\quad T_n:=\frac{(q;q^2)_{2n}}{(q^4;q^4)_{2n}}q^{4n(n+1)}$$

and compute the difference as follows:

$$\Delta T_n=-\frac{(q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}}q^{4n(n+1)}\Big\{1+q^{4n+1}\frac{(1-q^{4n+3})(1-q^{4n+7})}{(1-q^{8n+8})(1-q^{4n+1})}\Big\}.$$

Then applying (3) and replacing the summation index n by $n-1$, we find the following companion identity:

$$\frac{\left(q^{12},\ q,\ q^{11};q^{12}\right)_\infty}{(q^4;q^4)_\infty}=\sum_{n=0}^{\infty}\frac{(q^{-3};q^4)_n(q^7;q^4)_n}{(q^4;q^4)_{2n}}q^{4n^2}. \tag{42}$$

§17. **RR-identities related to modulo 12: B.** Analogously for the following identity displayed in Chu-Zhang [11, No 108]

$$\frac{\left(q^{12},\ q^3,\ q^9;q^{12}\right)_\infty}{(q^4;q^4)_\infty}=\sum_{n=0}^{\infty}\frac{(1-q^{4n+3})(q;q^2)_{2n}}{(q^4;q^4)_{1+2n}}q^{4n(n+1)} \tag{43}$$

the companion identity corresponding to (3) reads as

$$\frac{\left(q^{12},\ q^3,\ q^9;q^{12}\right)_\infty}{(q^4;q^4)_\infty}=\sum_{n=0}^{\infty}\frac{(q^{-1};q^2)_{2n}(q^5;q^4)_n}{(q^4;q^4)_{2n}(q;q^4)_n}q^{4n^2}. \tag{44}$$

This is confirmed by defining the sequence $\{T_n\}_{n\geq 0}$ through

$$T_0=1 \quad\text{and}\quad T_n:=\frac{(q;q^2)_{2n}}{(q^4;q^4)_{2n}}q^{4n(n+1)}$$

and then evaluating the following difference

$$\Delta T_n=-\frac{(1-q^{4n+3})(q;q^2)_{2n}}{(q^4;q^4)_{2n+1}}q^{4n(n+1)}\Big\{1+q^{4n+3}\frac{(1-q^{4n+1})(1-q^{4n+5})}{(1-q^{8n+8})(1-q^{4n+3})}\Big\}.$$

§18. **RR-identities related to modulo 12: C.** Rewrite an identity due to Bailey [8, Eq 7.6]

$$\frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty}\left(q^{12},q,q^{11};q^{12}\right)_\infty=\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{3n+1}\,q^{3n^2}}{(q^3;q^3)_{2n}(q^{12};q^{12})_n(q^{6n}-q^2)} \tag{45}$$

$$=1-\sum_{n=0}^{\infty}\frac{(1-q^{6n+8})(q^2;q^2)_{3n+1}}{(q^3;q^3)_{2n+1}(q^{12};q^{12})_{n+1}}q^{3n^2+6n+1}.$$

For the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=-1 \quad\text{and}\quad T_n:=-\frac{(q^2;q^2)_{3n}\,q^{3n^2+6n}}{(q^3;q^3)_{2n}(q^{12};q^{12})_n}$$

the corresponding forward difference reads as

$$\Delta T_n=q^{3n^2+6n+1}\frac{(1-q^{6n+8})\,(q^2;q^2)_{3n+1}}{(q^3;q^3)_{2n+1}(q^{12};q^{12})_{n+1}}\Big\{1-\frac{(1-q^{-1})(1-q^{12n+12})}{(1-q^{6n+2})(1-q^{6n+8})}\Big\}.$$

We therefore derive from (3) the following elegant RR-identity:

$$\frac{(-q^3;q^6)_\infty}{(q^6;q^6)_\infty}\left(q^{12},q^{11},q^{13};q^{12}\right)_\infty=\sum_{n=0}^{\infty}\frac{(q^2;q^2)_{3n}\,q^{3n^2+6n}}{(q^{12};q^{12})_n(q^3;q^3)_{2n+1}}. \tag{46}$$

§19. **RR-identities related to modulo 12: D.** Similarly, the combination of the identity displayed in Chu-Zhang [11, No 120]

$$\frac{(-q^6;q^6)_\infty}{(q^6;q^6)_\infty}\left(q^{12},\ q,\ q^{11};q^{12}\right)_\infty=\sum_{n=0}^{\infty}\frac{(q;q^2)_{2+3n}(-q^6;q^6)_n}{(q^6;q^6)_{1+2n}(q^3;q^6)_{1+n}}q^{3n(n+1)} \tag{47}$$

with the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=1 \quad\text{and}\quad T_n:=\frac{(q;q^2)_{3n}(-q^6;q^6)_n}{(q^6;q^6)_{2n}(q^3;q^6)_n}q^{3n(n+1)}$$

and its forward difference

$$\Delta T_n=-\frac{(q;q^2)_{3n+2}(-q^6;q^6)_n}{(q^6;q^6)_{2n+1}(q^3;q^6)_{n+1}}q^{3n(n+1)}\Big\{1+q^{6n+1}\frac{(1-q^5)(1-q^{6n+5})}{(1-q^{6n+6})(1-q^{6n+1})}\Big\}.$$

yields the following new RR-identity

$$\frac{(-q^6;q^6)_\infty}{(q^6;q^6)_\infty}\left(q^{12},q,q^{11};q^{12}\right)_\infty=\sum_{n=0}^{\infty}\frac{(q^{-5};q^6)_n(q^5;q^6)_n}{(q^6;q^6)_n(q^6;q^{12})_n}q^{3n(n+1)}. \tag{48}$$

§20. **RR-identities related to modulo 16: A.** In a recent paper, Sills [23, Eq 5.5] found the following identity

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(q^{16},\ q^8,\ q^8;q^{16}\right)_\infty=\sum_{n=0}^{\infty}\frac{(-q^2;q^4)_n\,q^{n^2}}{(q;q^2)_n(q^4;q^4)_n}. \tag{49}$$

By means of (3), we get its companion identity [11, No 133]

$$\frac{(-q^3;q^2)_\infty}{(q^2;q^2)_\infty}\left(q^{16},\ q^8,\ q^8;q^{16}\right)_\infty=\sum_{n=0}^{\infty}\frac{(-q;q^2)_n(-q^{-2};q^4)_n\,q^{n(n+2)}}{(-q^{-1};q^2)_n(q;q^2)_n(q^4;q^4)_n} \tag{50}$$

where we have the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=0 \quad\text{and}\quad T_n:=\frac{(-q^2;q^4)_{n-1}q^{n^2}}{(q;q^2)_{n-1}(q^4;q^4)_{n-1}}$$

with the forward difference given by

$$\Delta T_n=\frac{(-q^2;q^4)_nq^{n^2}}{(q;q^2)_n(q^4;q^4)_n}\Big\{q^{2n-1}\frac{(1+q^2)(1+q^{2n-1})}{(1+q^{4n-2})}-1\Big\}.$$

§21. **RR-identities related to modulo 16: B.** Analogously for the following identity due to Slater [24, Eq 70])

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(q^{16},\ q^4,\ q^{12};q^{16}\right)_\infty=\sum_{n=0}^\infty\frac{(-q^2;q^4)_n\ q^{n(n+2)}}{(q;q^2)_{n+1}(q^4;q^4)_n}\tag{51}$$

the companion identity corresponding to (3) reads as

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\left(q^{16},q^4,q^{12};q^{16}\right)_\infty=\sum_{n=0}^\infty\frac{(-q^{-2};q^4)_n\ q^{n(n+2)}}{(q;q^2)_n(q^4;q^4)_n}.\tag{52}$$

This is confirmed by defining the sequence $\{T_n\}_{n\geq 0}$ through

$$T_0=1\quad\text{and}\quad T_n:=\frac{(-q^2;q^4)_n\ q^{n(n+2)}}{(q;q^2)_n(q^4;q^4)_n}$$

and then evaluating the following difference

$$\Delta T_n=\frac{(-q^2;q^4)_n\ q^{n(n+2)}}{(q;q^2)_{n+1}(q^4;q^4)_n}\Big\{\frac{q^{2n+1}(1+q^2)}{1-q^{4n+4}}-1\Big\}.$$

§22. **RR-identities related to modulo 18.** Recall the identity displayed in Chu-Zhang [11, No 134]

$$\frac{\left(q^{18},\ q,\ q^{17};q^{18}\right)_\infty}{(q^6;q^6)_\infty}=\sum_{n=0}^\infty\frac{(q;q^2)_{3n+2}q^{6n(n+1)}}{(q^6;q^6)_{2n+1}(q^3;q^6)_{n+1}}.\tag{53}$$

Define the sequence $\{T_n\}_{n\geq 0}$ by

$$T_0=1\quad\text{and}\quad T_n:=\frac{(q;q^2)_{3n}q^{6n(n+1)}}{(q^6;q^6)_{2n}(q^3;q^6)_n}$$

where it is not difficult to verify that

$$\Delta T_n=-\frac{(q;q^2)_{3n+2}q^{6n(n+1)}}{(q^6;q^6)_{2n+1}(q^3;q^6)_{n+1}}\Big\{1+q^{6n+1}\frac{(1-q^{6n+5})(1-q^{6n+11})}{(1-q^{6n+1})(1-q^{12n+12})}\Big\}.$$

Then applying (3), we get the following equality

$$\frac{\left(q^{18},\ q,\ q^{17};q^{18}\right)_\infty}{(q^6;q^6)_\infty}=1-\sum_{n=0}^\infty\frac{(1-q^{6n+11})(q;q^2)_{3n+3}q^{6n^2+12n+1}}{(1-q^{6n+1})(q^6;q^6)_{2n+2}(q^3;q^6)_{n+1}}$$

which can be reformulated, under the replacement $n\to n-1$, to the new RR-identity:

$$\frac{\left(q^{18},\ q,\ q^{17};q^{18}\right)_\infty}{(q^6;q^6)_\infty}=\sum_{n=0}^\infty\frac{(q^{-5};q^6)_n(q^{11};q^6)_n}{(q^6;q^6)_{2n}}q^{6n^2}.\tag{54}$$

§23. **RR-identities related to moduli 6 and 12.** Observe that the following identity is equivalent to McLaughlin-Sills [16, Eq 1.24]:

$$\frac{(-q;q)_\infty}{(q;q)_\infty}\left(q^6,q^2,q^4;q^6\right)_\infty\left(q^{10},q^2;q^{12}\right)_\infty=\sum_{n=0}^\infty\frac{(-q^3;q^3)_n}{(q;q)_{2n+1}}q^{\binom{n+1}{2}}.\tag{55}$$

Then the telescopic approach leads us to the identity [11, No 162]

$$\frac{(-q^2;q)_\infty}{(q;q)_\infty}\left(q^6,q^2,q^4;q^6\right)_\infty\left(q^{10},q^2;q^{12}\right)_\infty=\sum_{n=0}^\infty\frac{(-q;q)_n(-1;q^3)_n}{(-q^{-1};q)_n(q;q)_{2n+1}}q^{\binom{n}{2}+2n}\tag{56}$$

in accordance with the sequence $\{T_n\}_{n\geq 0}$ given by

$$T_0=0\quad\text{and}\quad T_n:=\frac{(-q^3;q^3)_{n-1}}{(q;q)_{2n-1}}q^{\binom{n+1}{2}}$$

and the corresponding difference

$$\Delta T_n=q^{\binom{n+1}{2}}\frac{(-q^3;q^3)_n}{(q;q)_{2n+1}}\Big\{q^{n+1}\frac{(1+q^{n-1})(1+q^n)}{1+q^{3n}}-1\Big\}.$$

§24. **RR-identities related to moduli 9 and 18.** Note that an identity due to McLaughlin-Sills [16, Eq 1.7] may be expressed as

$$\frac{\big(q^9,-q,-q^8;q^9\big)_\infty}{(q;q)_\infty}\big(q^7,\,q^{11};q^{18}\big)_\infty=1+\sum_{n=1}^{\infty}\frac{(2+q^n)(q^3;q^3)_{n-1}}{(q;q)_{n-1}(q;q)_{2n}}q^{n^2} \tag{57}$$

$$=1+\sum_{n=0}^{\infty}\frac{(2+q^{n+1})(q^3;q^3)_n}{(q;q)_n(q;q)_{2n+2}}q^{(n+1)^2}.$$

For the sequence $\{T_n\}_{n\geq 0}$ defined by

$$T_0=-1\quad\text{and}\quad T_n:=-\frac{(q^3;q^3)_n\,q^{n^2}}{(q;q)_n(q;q)_{2n}}$$

it is not hard to verify that

$$\Delta T_n=\frac{(q^3;q^3)_n\,q^{n^2}}{(q;q)_n(q;q)_{2n+2}}\Big\{(1-q^{2n+2})-q^{2n+1}(2+q^{n+1})\Big\}.$$

Then the equation corresponding to (3) yields the following companion identity:

$$\frac{\big(q^9,-q,-q^8;q^9\big)_\infty}{(q;q)_\infty}\big(q^7,\,q^{11};q^{18}\big)_\infty=\sum_{n=0}^{\infty}\frac{(q^3;q^3)_n\,q^{n^2}}{(q;q)_n(q;q)_{2n+1}}. \tag{58}$$

§25. **RR-identities related to moduli 12 and 24.** Finally reformulate an identity due to McLaughlin-Sills [16, Eq 1.17]

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big(q^{12},-q,-q^{11};q^{12}\big)_\infty\big(q^{10},q^{14};q^{24}\big)_\infty \tag{59a}$$

$$=\quad 1+\sum_{n=1}^{\infty}\frac{(2+q^{2n})(-q;q^2)_n(q^6;q^6)_{n-1}}{(q^2;q^2)_{2n}(q^2;q^2)_{n-1}}q^{n^2} \tag{59b}$$

$$=\quad 1+\sum_{n=0}^{\infty}\frac{(2+q^{2n+2})(-q;q^2)_{n+1}(q^6;q^6)_n}{(q^2;q^2)_{2n+2}(q^2;q^2)_n}q^{(n+1)^2}.$$

We recover through (3) the nicer looking identity due to Chu-Zhang [11, No 181]

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty}\big(q^{12},-q,-q^{11};q^{12}\big)_\infty\big(q^{10},q^{14};q^{24}\big)_\infty=\sum_{n=0}^{\infty}\frac{(-q;q^2)_{n+1}(q^6;q^6)_n}{(q^2;q^2)_{2n+1}(q^2;q^2)_n}q^{n^2}. \tag{60}$$

This is done by choosing the sequence $\{T_n\}_{n\geq 0}$ with

$$T_0=-1\quad\text{and}\quad T_n:=-\frac{(-q;q^2)_n(q^6;q^6)_n}{(q^2;q^2)_{2n}(q^2;q^2)_n}q^{n^2}$$

and verifying the corresponding forward difference

$$\Delta T_n=q^{n^2}\frac{(-q;q^2)_{n+1}(q^6;q^6)_n}{(q^2;q^2)_{2n+2}(q^2;q^2)_n}\Big\{(1-q^{4n+4})-q^{2n+1}(2+q^{2n+2})\Big\}.$$

Concluding comments We believe that the examples just exhibited have served as enough evidence for the efficiency of telescopic approach. There may exist numerous other RR-identities fitting into this creation scheme. However, we shall not attempt to make a full coverage due to the space limitation.

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