Finite Differences and Dixon–Like Binomial Sums [†]

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Summary Finite difference method is employed to investigate Dixon–like binomial sums. Several identities are established which extend the alternating cubic sums of Dixon and Vosmansky.

1. Introduction and Motivation

Let Δ be the usual difference operator with the unit increment. For a real or complex function $f(\tau)$, the finite differences of order n can be calculated through the following Newton–Gregory formula (cf. Graham *et al* [7, §5.3])

$$\Delta^n f(\tau) = \sum_{k=0}^{n} (-1)^{n+k} \binom{n}{k} f(\tau + k). \tag{1}$$

When $f(\tau)$ is a polynomial of degree $m \le n$, then $\Delta^n f(\tau)$ vanishes for $0 \le m < n$ and otherwise, equals $m!$ times the leading coefficient of $f(\tau)$ for $m = n$. These useful properties have recently been utilized by the author [3, 4] to evaluate several Hankel determinants and to give elementary proofs for the convolution identities of Abel and Hagen–Rothe.

By employing the finite difference method further, we shall investigate the alternating binomial sums of the following form

$$U_n(\lambda, \varepsilon | y) := \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+y}{n+\varepsilon} \binom{k-y+\lambda}{n+\varepsilon} \tag{2}$$

where $\lambda, \varepsilon \in \mathbb{Z}$, $n \in \mathbb{N}_0$ and y is an indeterminate. This has partially been motivated by the recent work due to Gould–Quaintance [6], who obtained a closed formula for the case $n = 2m$ and $\varepsilon = 1 + \lambda$, extending an earlier result found by Vosmansky [9]. The the binomial sum displayed in (2) will be said "Dixon–like" because in terms of hypergeometric series (see section 5), it becomes "almost–poised", just like the Dixon–sum, which is well–poised ${}_3F_2$-series.

The polynomial $U_n(\lambda, \varepsilon|y)$ is of even degree $\le n + 2\varepsilon$ in variable y. This is disguised in the binomial sum (2), even though it will be almost evident to see from another expression (3). Because the strategy for us to find summation formulae will start from (2), it is necessary to determine universally the precise degree of $U_n(\lambda, \varepsilon|y)$ as a polynomial of y in an independent manner.

This can be done by finite differences. Replacing y by $x + \lambda/2$, we have

$$U_n(\lambda, \varepsilon | x + \lambda/2) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+x+\frac{\lambda}{2}}{n+\varepsilon} \binom{k-x+\frac{\lambda}{2}}{n+\varepsilon}$$

which is an even function in x and can consequently be considered as a polynomial in x^2. Observe that the coefficient of $x^{2\ell}$ in $\binom{k+x+\frac{\lambda}{2}}{n+\varepsilon}\binom{k-x+\frac{\lambda}{2}}{n+\varepsilon}$ is a polynomial of degree $2(n+\varepsilon-\ell)$ in k that will be annihilated by the finite differences of order

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n when $\ell > \varepsilon + \frac{n}{2}$ because the polynomial degree $2(n+\varepsilon-\ell)$ is, in this case, less than the difference order n. Therefore $U_n(\lambda,\varepsilon|x+\lambda/2)$ results in a polynomial of even degree $\le n+2\varepsilon$ in x which confirms that $U_n(\lambda,\varepsilon|y)$ has also the same even degree $\le n+2\varepsilon$ in the variable y.

According to the fundamental theorem of algebra, we shall use the following algorithm to prove the binomial identities presented throughout the paper.

- Let $\Omega(y)$ denote the binomial sum. Find out the polynomial degree of $\Omega(y)$ and the zeros of $\Omega(y)$.
- By means of informed observation, figure out $\omega(y)$, which is also a polynomial in y with the same degree and the same zeros as $\Omega(y)$.
- Determine the constant β such that $\beta\omega(y)$ results in the desired closed form for our summation $\Omega(y)$.

We illustrate this procedure by offering the following new proof of the binomial identity below found by Gould–Quaintance [6, Theorem 2.2] through *the creative telescoping algorithm* of Zeilberger [8].

Theorem 1 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m}(\lambda,\lambda+1|y) &= \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+\lambda+1}\binom{k-y+\lambda}{2m+\lambda+1}\\ &= \frac{\binom{2m}{m}}{\binom{1+2m+\lambda}{m}}\binom{y}{m+\lambda+1}\binom{\lambda-y}{m+\lambda+1}. \end{aligned}$$

When $\lambda=-1$, this theorem recovers the identity found by Vosmansky [9]. By means of the generating function method, Carlitz *et al* [2] presented a direct proof.

Proof. Define the polynomial by the binomial sum on the left hand side

$$F(y)=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+\lambda+1}\binom{k-y+\lambda}{2m+\lambda+1}.$$

This is a polynomial of degree $2+2m+2\lambda$ in y. In order to evaluate $F(y)$, we first examine the case $\lambda<0$. For $i=0,1,\cdots,m+\lambda$, we assert that $F(i)=0$ because each binomial term appeared in $F(i)$ is equal to zero. Otherwise, we would simultaneously have both $1+2m+\lambda \le k+i$ and $k-i+\lambda<0$ which lead to $i>m+\lambda$. In view of the symmetry $F(y)=F(\lambda-y)$, we find that all the zeros of $F(y)$ are given by $\{i,\lambda-i : 0\le i\le m+\lambda\}$. Observing further that the binomial product $\binom{y}{m+\lambda+1}\binom{\lambda-y}{m+\lambda+1}$ has the same zeros as $F(y)$, there must exist a constant β such that

$$F(y)=\beta\binom{y}{m+\lambda+1}\binom{\lambda-y}{m+\lambda+1} \quad\text{with}\quad \beta=\frac{\binom{2m}{m}}{\binom{1+2m+\lambda}{m}}$$

where β has been determined by letting $y=1+m+\lambda$ in the last equation which is facilitated by the fact that there is only one term corresponding to $k=m$ survived in the binomial sum $F(1+m+\lambda)$.

When $\lambda\ge 0$, consider similarly the fraction $\mathcal{F}(y)=\frac{F(y)}{\binom{y}{\lambda+1}\binom{\lambda-y}{\lambda+1}}$. Following the same procedure as that for $F(y)$, we can show that $\mathcal{F}(y)$ is a polynomial of degree $2m$

with all the zeros being $\{i, \lambda - i : 1 + \lambda \le i \le m + \lambda\}$. Determining the constant factor at the same point $y = 1 + m + \lambda$, we arrive at the identity displayed in Theorem 1 also when $\lambda > 0$.

Summing up, we have confirmed the binomial identity stated in Theorem 1. $\square$

In the next two sections, we are going to establish ten further similar identities for the binomial sums displayed in (2) that are equally distributed in Sections 2 and 3 according to the parity of n. Except for Theorem 9 which corresponds to the case $\varepsilon = 1 + \lambda$, all the other nine identities are believed to be new because they do not belong to the classical hierarchy of well–poised hypergeometric series. Then in the fourth section, we shall transform (2), by means of the Leibniz rule of finite differences for the product of two functions, into another class of binomial sums

$$V_n(\lambda, \varepsilon|y) := \sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\lambda - y}{k + \varepsilon}\binom{\varepsilon - 1 - y}{\varepsilon + n - k} \tag{3}$$

which enables us consequently to deduce further eleven closed formulae including Dixon's well–known one for the alternating cubic binomial sum. Finally in the fifth section, the paper will end with a discussion on the connection between the classical hypergeometric series and the cubic binomial sums treated in the present paper.

2. Alternating Binomial Sums: $n = 2m$

This section will present two closed formulae for the binomial sum displayed in (2) with $\varepsilon - \lambda = 0, 2$ and three exceptional identities without λ-parameter. Because it suffices to carry out the proving procedure described in the introduction, we are limited to sketch only the key steps.

Theorem 2 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m}(\lambda, \lambda|y) &= \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k + y}{2m + \lambda}\binom{k - y + \lambda}{2m + \lambda} \\ &= \frac{\binom{2m}{m}}{\binom{2m+\lambda}{m}}\binom{y - 1}{m + \lambda}\binom{\lambda - y - 1}{m + \lambda}. \end{aligned}$$

Sketch of proof. When $\lambda \le 0$, this theorem can be shown by defining

$$A(y) = \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k + y}{2m + \lambda}\binom{k - y + \lambda}{2m + \lambda}$$

and then verifying the following statements:

- $A(y)$ is a polynomial of degree $2m + 2\lambda$.
- All the zeros of $A(y)$ are $\{i, \lambda - i : 1 \le i \le m + \lambda\}$.
- The constant factor can be determined at $y = 0$.

Alternatively for $\lambda > 0$, Theorem 2 can be confirmed by defining the polynomial $\mathcal{A}(y) = \frac{A(y)}{\binom{y-1}{\lambda-1}\binom{\lambda-y-1}{\lambda-1}}$ and then proving the following three statements:

- $\mathcal{A}(y)$ is a polynomial of degree $2m + 2$.
- All the zeros of $\mathcal{A}(y)$ are $\{i, \lambda - i : \lambda \le i \le m + \lambda\}$.
- The constant factor can be determined at the same point $y = 0$.

Theorem 3 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m}(\lambda,\lambda+2|y) &= \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+\lambda+2}\binom{k-y+\lambda}{2m+\lambda+2} \\ &= \frac{\binom{2m}{m}}{\binom{2+2m+\lambda}{m}}\binom{y}{m+\lambda+2}\binom{\lambda-y}{m+\lambda+2}. \end{aligned}$$

Sketch of proof. When $\lambda<0$, this theorem can be shown by defining

$$B(y)=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+\lambda+2}\binom{k-y+\lambda}{2m+\lambda+2}$$

and then verifying the following statements:

- $B(y)$ is a polynomial of degree $2m+2\lambda+4$.
- All the zeros of $B(y)$ are $\{i,\lambda-i: 0\le i\le 1+m+\lambda\}$.
- The constant factor can be determined at $y=2+m+\lambda$.

Alternatively for $\lambda\ge 0$, Theorem 3 can be confirmed by defining the polynomial $\mathcal{B}(y)=\frac{B(y)}{\binom{y}{\lambda+2}\binom{\lambda-y}{\lambda+2}}$ and then proving the following three statements:

- $\mathcal{B}(y)$ is a polynomial of degree $2m$.
- All the zeros of $\mathcal{B}(y)$ are $\{i,\lambda-i: 2+\lambda\le i\le 1+m+\lambda\}$.
- The constant factor can be determined at the same point $y=2+m+\lambda$.

In addition, we have three exceptional formulae without the integer parameter λ.

Theorem 4 ($m \in \mathbb{N}_0$)**.**

$$\begin{aligned} U_{2m}(3,1|y) &= \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+1}\binom{k-y+3}{2m+1} \\ &= \frac{(2m+y)(3+2m-y)}{(m+1)(2m+1)}\binom{y-3}{m}\binom{-y}{m}. \end{aligned}$$

Sketch of proof. This theorem can be shown by defining

$$C(y)=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+1}\binom{k-y+3}{2m+1}$$

and then verifying the following statements:

- $C(y)$ is a polynomial of degree $2m+2$.
- All the zeros of $C(y)$ are $\{i,3-i: 3\le i\le 2+m\}$ plus $\{3+2m,-2m\}$.
- The constant factor can be determined at $y=2$.

Theorem 5 ($m \in \mathbb{N}_0$)**.**

$$\begin{aligned} U_{2m}(-3,1|y) &= \sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+1}\binom{k-y-3}{2m+1} \\ &= \frac{(2m-y)(3+2m+y)}{(m+1)(2m+1)}\binom{y}{m}\binom{-y-3}{m}. \end{aligned}$$

Sketch of proof. This theorem can be shown by defining

$$D(y)=\sum_{k=0}^{2m}(-1)^k\binom{2m}{k}\binom{k+y}{2m+1}\binom{k-y-3}{2m+1}$$

and then verifying the following statements:

- $D(y)$ is a polynomial of degree $2m+2$.
- All the zeros of $D(y)$ are $\{i, 3-i : 0 \le i \le m-1\}$ plus $\{2m, -3-2m\}$.
- The constant factor can be determined at $y=-1$.

Theorem 6 ($m \in \mathbb{N}_0$)**.**

$$\begin{aligned} U_{2m}(1,-1|y) &= \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{k+y}{2m-1} \binom{k-y+1}{2m-1} \\ &= \frac{m^2(2m-y)(2m-1+y)\binom{y-1}{m}\binom{-y}{m}}{12\binom{y+1}{4}}. \end{aligned}$$

Sketch of proof. This theorem can be shown by defining

$$E(y) = \sum_{k=0}^{2m} (-1)^k \binom{2m}{k} \binom{k+y}{2m-1} \binom{k-y+1}{2m-1}$$

and then verifying the following statements:

- $E(y)$ is a polynomial of degree $2m-2$.
- All the zeros of $E(y)$ are $\{i, 3-i : 3 \le i \le m\}$ plus $\{2m, 1-2m\}$.
- The constant factor can be determined at $y=2$.

3. Alternating Binomial Sums: $n = 2m+1$

When n is odd, we shall evaluate, in this section, five binomial sums defined in (2) for the integers λ and ε with $\varepsilon - \lambda = 0, \pm 1, 2, 3$. Now that the proving procedure is entirely the same as that for Theorem 1, we shall confine ourselves to sketch briefly the crucial steps, instead of producing details.

Theorem 7 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m+1}(\lambda, \lambda-1|y) &= \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \binom{k+y}{2m+\lambda} \binom{k-y+\lambda}{2m+\lambda} \\ &= -\frac{\binom{2m+2}{m+1}}{\binom{2m+\lambda}{m+1}} \binom{y-2}{m+\lambda-1} \binom{\lambda-y-2}{m+\lambda-1}. \end{aligned}$$

Sketch of proof. When $\lambda \le 0$, this theorem can be shown by defining

$$P(y) = \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \binom{k+y}{2m+\lambda} \binom{k-y+\lambda}{2m+\lambda}$$

and then verifying the following statements:

- $P(y)$ is a polynomial of degree $2m+2\lambda-2$.
- All the zeros of $P(y)$ are $\{i, \lambda-i : 2 \le i \le m+\lambda\}$.
- The constant factor can be determined at $y=1$.

Alternatively for $\lambda > 0$, Theorem 7 can be confirmed by defining the polynomial $\mathcal{P}(y) = \frac{P(y)}{\binom{y-2}{\lambda-1}\binom{\lambda-y-2}{\lambda-1}}$ and then proving the following three statements:

- $\mathcal{P}(y)$ is a polynomial of degree $2m$.
- All the zeros of $\mathcal{P}(y)$ are $\{i, \lambda-i : 1+\lambda \le i \le m+\lambda\}$.
- The constant factor can be determined at the same point $y=1$.

Theorem 8 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m+1}(\lambda,\lambda|y) &= \sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\binom{k+y}{2m+\lambda+1}\binom{k-y+\lambda}{2m+\lambda+1} \\ &= -\frac{\binom{2m+1}{m+1}}{\binom{1+2m+\lambda}{m+1}}\binom{y-1}{m+\lambda}\binom{\lambda-y-1}{m+\lambda}. \end{aligned}$$

Sketch of proof. When $\lambda \leq 0$, this theorem can be shown by defining

$$Q(y) = \sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\binom{k+y}{2m+\lambda+1}\binom{k-y+\lambda}{2m+\lambda+1}$$

and then verifying the following statements:

- $Q(y)$ is a polynomial of degree $2m+2\lambda$.
- All the zeros of $Q(y)$ are $\{i, \lambda-i : 1 \leq i \leq m+\lambda\}$.
- The constant factor can be determined at $y = 0$.

Alternatively for $\lambda > 0$, Theorem 8 can be confirmed by defining the polynomial $\mathcal{Q}(y) = \frac{Q(y)}{\binom{y-1}{\lambda}\binom{\lambda-y-1}{\lambda}}$ and then proving the following three statements:

- $\mathcal{Q}(y)$ is a polynomial of degree $2m$.
- All the zeros of $\mathcal{Q}(y)$ are $\{i, \lambda-i : 1+\lambda \leq i \leq m+\lambda\}$.
- The constant factor can be determined at the same point $y = 0$.

Theorem 9 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$U_{2m+1}(\lambda,\lambda+1|y) = \sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\binom{k+y}{2m+\lambda+2}\binom{k-y+\lambda}{2m+\lambda+2} = 0.$$

Sketch of proof. When $\lambda < 0$, this theorem can be shown by defining

$$R(y) = \sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\binom{k+y}{2m+\lambda+2}\binom{k-y+\lambda}{2m+\lambda+2}$$

and then verifying the following statements:

- $R(y)$ is a polynomial of degree $2m+2\lambda+2$.
- $R(y)$ has zeros $\{i, \lambda-i : 0 \leq i \leq m+\lambda+1\}$ whose cardinality is equal to $2m+2\lambda+4$, greater than the degree of $R(y)$; which forces $R(y) \equiv 0$.

Alternatively for $\lambda \geq 0$, Theorem 9 can be confirmed by defining the polynomial $\mathcal{R}(y) = \frac{R(y)}{\binom{y}{\lambda+1}\binom{\lambda-y}{\lambda+1}}$ and then proving the following three statements:

- $\mathcal{R}(y)$ is a polynomial of degree $2m$.
- $\mathcal{R}(y)$ has zeros $\{i, \lambda-i : 1+\lambda \leq i \leq m+\lambda+1\}$ whose cardinality is equal to $2m+2$, greater than the degree of $\mathcal{R}(y)$; which leads to $\mathcal{R}(y) \equiv 0$.

Theorem 10 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m+1}(\lambda,\lambda+2|y) &= \sum_{k=0}^{2m+1}(-1)^k\binom{2m+1}{k}\binom{k+y}{2m+\lambda+3}\binom{k-y+\lambda}{2m+\lambda+3} \\ &= \frac{\binom{2m+1}{m+1}}{\binom{3+2m+\lambda}{m+1}}\binom{y}{2+m+\lambda}\binom{\lambda-y}{2+m+\lambda}. \end{aligned}$$

Sketch of proof. When $\lambda < 0$, this theorem can be shown by defining

$$S(y) = \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \binom{k+y}{2m+\lambda+3} \binom{k-y+\lambda}{2m+\lambda+3}$$

and then verifying the following statements:

- $S(y)$ is a polynomial of degree $2m+2\lambda+4$.
- All the zeros of $S(y)$ are $\{i, \lambda - i : 0 \le i \le m+\lambda+1\}$.
- The constant factor can be determined at $y = m+\lambda+2$.

Alternatively for $\lambda \ge 0$, Theorem 10 can be confirmed by defining the polynomial $\mathcal{S}(y) = \frac{S(y)}{\binom{y}{\lambda+2}\binom{\lambda-y}{\lambda+2}}$ and then proving the following three statements:

- $\mathcal{S}(y)$ is a polynomial of degree $2m$.
- All the zeros of $\mathcal{S}(y)$ are $\{i, \lambda - i : 2+\lambda \le i \le m+\lambda+1\}$.
- The constant factor can be determined at the same point $y = m+\lambda+2$.

Theorem 11 ($m \in \mathbb{N}_0$ and $\lambda \in \mathbb{Z}$)**.**

$$\begin{aligned} U_{2m+1}(\lambda, \lambda+3|y) &= \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \binom{k+y}{2m+\lambda+4} \binom{k-y+\lambda}{2m+\lambda+4} \\ &= \frac{\binom{2m+2}{m+1}}{\binom{4+2m+\lambda}{m+1}} \binom{y}{3+m+\lambda} \binom{\lambda-y}{3+m+\lambda}. \end{aligned}$$

Sketch of proof. When $\lambda < 0$, this theorem can be shown by defining

$$T(y) = \sum_{k=0}^{2m+1} (-1)^k \binom{2m+1}{k} \binom{k+y}{2m+\lambda+4} \binom{k-y+\lambda}{2m+\lambda+4}$$

and then verifying the following statements:

- $T(y)$ is a polynomial of degree $2m+2\lambda+6$.
- All the zeros of $T(y)$ are $\{i, \lambda - i : 0 \le i \le m+\lambda+2\}$.
- The constant factor can be determined at $y = m+\lambda+3$.

Alternatively for $\lambda \ge 0$, Theorem 11 can be confirmed by defining the polynomial $\mathcal{T}(y) = \frac{T(y)}{\binom{y}{\lambda+3}\binom{\lambda-y}{\lambda+3}}$ and then proving the following three statements:

- $\mathcal{T}(y)$ is a polynomial of degree $2m$.
- All the zeros of $\mathcal{T}(y)$ are $\{i, \lambda - i : 3+\lambda \le i \le m+\lambda+2\}$.
- The constant factor can be determined at the same point $y = m+\lambda+3$.

4. Leibniz Rule and Dixon–Like Formulae

Recall the Leibniz rule for the product of two functions

$$\Delta^n \big\{ f(\tau) g(\tau) \big\} = \sum_{k=0}^{n} \binom{n}{k} \Delta^k f(\tau) \Delta^{n-k} g(\tau+k).$$

Combining the binomial relation

$$\Delta^k \binom{x+\tau}{m} = \binom{x+\tau}{m-k}$$

with the difference expression

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{k+x}{p} \binom{k+y}{q} = (-1)^n \Delta^n \Big\{ \binom{x+\tau}{p} \binom{y+\tau}{q} \Big\}_{\tau=0}$$

we derive the following binomial transformation theorem.

Proposition 12 (Transformation on alternating binomial sums)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+x}{p}\binom{k+y}{q}=\sum_{k=0}^{n}(-1)^n\binom{n}{k}\binom{x}{p-k}\binom{y+k}{q+k-n}.$$

This generalizes slightly the transformation formula due to Gould and Quaintance [6, Theorem 3.1] from four to five free parameters. In particular, for the cubic sum displayed in (2), the corresponding transformation reads as

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+y}{n+\varepsilon}\binom{k-y+\lambda}{n+\varepsilon}&=\sum_{k=0}^{n}(-1)^n\binom{n}{k}\binom{k+y}{k+\varepsilon}\binom{\lambda-y}{n-k+\varepsilon}\\&=\sum_{k=0}^{n}(-1)^n\binom{n}{k}\binom{\lambda-y}{k+\varepsilon}\binom{n-k+y}{n-k+\varepsilon}\\&=\sum_{k=0}^{n}(-1)^{k+\varepsilon}\binom{n}{k}\binom{\lambda-y}{k+\varepsilon}\binom{\varepsilon-1-y}{n-k+\varepsilon}\end{aligned}$$

where the sum in the middle is justified by the replacement $k\to n-k$ and that one in the ultimate line by inverting the rightmost binomial coefficient. Connecting the first and the last sums, we can further reformulate the resulting equality as

$$U_n(\lambda,\varepsilon|y)=(-1)^{\varepsilon}V_n(\lambda,\varepsilon|y)$$

which can explicitly be stated as the following relation.

Corollary 13 (Transformation on alternating binomial sums)**.**

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+y}{n+\varepsilon}\binom{k-y+\lambda}{n+\varepsilon}=\sum_{k=0}^{n}(-1)^{k+\varepsilon}\binom{n}{k}\binom{\lambda-y}{k+\varepsilon}\binom{\varepsilon-1-y}{\varepsilon+n-k}.$$

For the cubic binomial sum, Dixon's identity (cf. Comtet [5, P174] and Graham *et al* [7, §5.5]) is well–known

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^3=\begin{cases}0, & n-\text{odd};\\(-1)^m\frac{(3m)!}{(m!)^3}, & n=2m.\end{cases}$$

When $1+\lambda=\varepsilon=0$ and $y=-1-n$, the binomial sum on the right hand side displayed in the equation of Corollary 13 becomes the last cubic binomial sum. This encourages us to transform all the summation formulae exhibited in the last two sections into Dixon–like identities. In order for the readers to have an easy access, they are collected in the following table, where the "Note" for each formula indicates the theorem based on which the formula has been derived.

n	λ	ε	Closed Expression for $V_n(\lambda,\varepsilon\|y)$	Note
$2m$	λ	$1+\lambda$	$\frac{(-1)^{1+\lambda}\binom{2m}{m}}{\binom{1+2m+\lambda}{m}}\dbinom{y}{m+\lambda+1}\dbinom{\lambda-y}{m+\lambda+1}$	Theorem 1
$2m$	λ	λ	$\frac{(-1)^{\lambda}\binom{2m}{m}}{\binom{2m+\lambda}{m}}\dbinom{y-1}{m+\lambda}\dbinom{\lambda-y-1}{m+\lambda}$	Theorem 2
$2m$	λ	$2+\lambda$	$\frac{(-1)^{\lambda}\binom{2m}{m}}{\binom{2+2m+\lambda}{m}}\dbinom{y}{m+\lambda+2}\dbinom{\lambda-y}{m+\lambda+2}$	Theorem 3
$2m$	3	1	$\frac{(y+2m)(y-2m-3)}{(m+1)(2m+1)}\dbinom{y-3}{m}\dbinom{-y}{m}$	Theorem 4
$2m$	-3	1	$\frac{(y-2m)(y+2m+3)}{(m+1)(2m+1)}\dbinom{y}{m}\dbinom{-y-3}{m}$	Theorem 5
$2m$	1	-1	$\frac{m^2(y-2m)(y+2m-1)\binom{y-1}{m}\binom{-y}{m}}{12\binom{y+1}{4}}$	Theorem 6
$2m+1$	λ	$\lambda-1$	$\frac{(-1)^{\lambda}\binom{2m+2}{m+1}}{\binom{2m+\lambda}{m+1}}\dbinom{y-2}{m+\lambda-1}\dbinom{\lambda-y-2}{m+\lambda-1}$	Theorem 7
$2m+1$	λ	λ	$\frac{(-1)^{1+\lambda}\binom{2m+1}{m+1}}{\binom{1+2m+\lambda}{m+1}}\dbinom{y-1}{m+\lambda}\dbinom{\lambda-y-1}{m+\lambda}$	Theorem 8
$2m+1$	λ	$1+\lambda$	0	Theorem 9
$2m+1$	λ	$2+\lambda$	$\frac{(-1)^{\lambda}\binom{2m+1}{m+1}}{\binom{3+2m+\lambda}{m+1}}\dbinom{y}{2+m+\lambda}\dbinom{\lambda-y}{2+m+\lambda}$	Theorem 10
$2m+1$	λ	$3+\lambda$	$\frac{(-1)^{1+\lambda}\binom{2m+2}{m+1}}{\binom{4+2m+\lambda}{m+1}}\dbinom{y}{3+m+\lambda}\dbinom{\lambda-y}{3+m+\lambda}$	Theorem 11

5. Terminating Almost–Poised ${}_3F_2$-Series

For an indeterminate x and a natural number n, denote the shifted–factorial by

$$(x)_0 = 1 \quad\text{and}\quad (x)_n = x(x+1)\cdots(x+n-1) \quad\text{where}\quad n\in\mathbb{N}.$$

Following Bailey [1], the generalized hypergeometric series ${}_{1+r}F_r$, for an indeterminate z and a nonnegative integer r, is defined by

$${}_{1+r}F_r\left[\begin{matrix}a_0, a_1, \cdots, a_r\\ b_1, \cdots, b_r\end{matrix}\,\middle|\, z\right] = \sum_{k=0}^{\infty}\frac{(a_0)_k(a_1)_k\cdots(a_r)_k}{k!(b_1)_k\cdots(b_r)_k}z^k.$$

When one of numerator parameters $\{a_k\}$ is a negative integer, then the series becomes terminating, which reduces to a polynomial in z. In particular if the parameters satisfy the condition $1+a_0 = a_1+b_1 = \cdots = a_r+b_r$, then the series is said to be well–poised, which has been well–studied in the classical theory of hypergeometric series.

In terms of hypergeometric series, both binomial sums $U_n(\lambda,\varepsilon|y)$ and $V_n(\lambda,\varepsilon|y)$ displayed respectively in (2) and (3) can be expressed as

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{k+y}{n+\varepsilon}\binom{k-y+\lambda}{n+\varepsilon}=\binom{y}{n+\varepsilon}\binom{\lambda-y}{n+\varepsilon}$$
$$\times\,{}_3F_2\left[\begin{matrix}-n, & 1+y, & 1+\lambda-y\\ 1+\lambda-y-n-\varepsilon, & 1+y-n-\varepsilon\end{matrix}\,\middle|\,1\right],$$
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{\lambda-y}{k+\varepsilon}\binom{\varepsilon-1-y}{\varepsilon+n-k}=\binom{\lambda-y}{\varepsilon}\binom{\varepsilon-1-y}{n+\varepsilon}(-1)^{\varepsilon}$$
$$\times\,{}_3F_2\left[\begin{matrix}-n, \varepsilon-\lambda+y, -n-\varepsilon\\ -y-n, \quad 1+\varepsilon\end{matrix}\,\middle|\,1\right].$$

When $\varepsilon = 1+\lambda$, both ${}_3F_2$-series are well–poised, which fall into the classical hierarchy of terminating well–poised ${}_3F_2$-series. When $\varepsilon \neq 1+\lambda$, they are called "almost–poised" series that do not have, in general, closed forms. Therefore from this point of view, the formulae presented in this paper are particularly remarkable.

Before concluding the paper, we remark that it is generally not an easy task to figure out all the zeros for the polynomial $U_n(\lambda,\varepsilon,|y)$, which need good fortune and coincidence. For example, the additional polynomial zeros in the proofs of Theorems 4, 5 and 6 have been justified by the formulae of Watson and Whipple for ${}_3F_2$-series (cf. Bailey [1, §3.3 and §3.4]).

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