

ANALYTICAL FORMULAE FOR EXTENDED  
 ${}_3F_2$ -SERIES OF WATSON–WHIPPLE–DIXON  
 WITH TWO EXTRA INTEGER PARAMETERS †

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**Summary** By combining the linearization method with Dougall’s sum for well-poised  ${}_5F_4$ -series, we investigate the generalized Watson series with two extra integer parameters. Four analytical formulae are established, which can be used to evaluate also the extended Whipple and Dixon series via the Thomae transformation. Twelve concrete formulae are presented as exemplification.

1. INTRODUCTION AND MOTIVATION

For an indeterminate  $x$  and a natural number  $n$ , denote the shifted-factorials by

$$\begin{aligned} (x)_0 &= 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for} \quad n \in \mathbb{N}; \\ \langle x \rangle_0 &= 1 \quad \text{and} \quad \langle x \rangle_n = x(x-1)\cdots(x-n+1) \quad \text{for} \quad n \in \mathbb{N}. \end{aligned}$$

Following Bailey [1], the hypergeometric series, for an indeterminate  $z$  and two nonnegative integers  $p$  and  $q$ , is defined by

$${}_1+{}_pF_q \left[ \begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_p)_k}{k! (b_1)_k \cdots (b_q)_k} z^k$$

where  $\{a_i\}$  and  $\{b_j\}$  are complex parameters such that no zero factors appear in the denominators of the summands on the right hand side. For the sake of brevity, the product and fraction of shifted factorials will be abbreviated respectively as

$$\begin{aligned} [A, B, \dots, C]_n &= (A)_n (B)_n \cdots (C)_n, \\ \left[ \begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_n &= \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n}. \end{aligned}$$

The  $\Gamma$ -function quotient will analogously be shortened as

$$\Gamma \left[ \begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right] = \frac{\Gamma(\alpha)\Gamma(\beta)\cdots\Gamma(\gamma)}{\Gamma(A)\Gamma(B)\cdots\Gamma(C)}.$$

There are many hypergeometric series formulae in the literature of mathematics and physics. Among them, the following three formulae for nonterminating series have been fundamental in the theory of classical hypergeometric series (cf. Bailey [1, Chapter III] and Chu [5]).

- Dixon [6]:  $\Re(1 + \frac{a}{2} - b - c) > 0$

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ 1+a-b, 1+a-c \end{matrix} \middle| 1 \right] = \Gamma \left[ \begin{matrix} 1+a-b, 1+a-c, 1+\frac{a}{2}, 1+\frac{a}{2}-b-c \\ 1+\frac{a}{2}-b, 1+\frac{a}{2}-c, 1+a, 1+a-b-c \end{matrix} \right]. \quad (1)$$

- Watson [19]:  $\Re(1 - a - b + 2c) > 0$

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ \frac{1+a+b}{2}, 2c \end{matrix} \middle| 1 \right] = \Gamma \left[ \begin{matrix} \frac{1}{2}, \frac{1+a+b}{2}, \frac{1}{2}+c, \frac{1-a-b}{2}+c \\ \frac{1+a}{2}, \frac{1+b}{2}, \frac{1-a}{2}+c, \frac{1-b}{2}+c \end{matrix} \right]. \quad (2)$$

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- Whipple [20]:  $\Re(c) > 0$

$${}_3F_2 \left[ \begin{matrix} a, 1-a, c \\ e, 1+2c-e \end{matrix} \middle| 1 \right] = \Gamma \left[ \begin{matrix} \frac{e}{2}, \frac{1+e}{2}, c + \frac{1-e}{2}, c + \frac{2-e}{2} \\ \frac{e+a}{2}, \frac{1+e-a}{2}, c + \frac{1-e+a}{2}, c + \frac{2-e-a}{2} \end{matrix} \right]. \quad (3)$$

As shown by Bailey [1, Chapter III], these three  ${}_3F_2(1)$ -series are connected by the following useful transformation due to Kummer–Thomae–Whipple, which can be reproduced as

$${}_3F_2 \left[ \begin{matrix} a, c, e \\ b, d \end{matrix} \middle| 1 \right] = \Gamma \left[ \begin{matrix} b, d, \lambda \\ a, c+\lambda, e+\lambda \end{matrix} \right] {}_3F_2 \left[ \begin{matrix} b-a, d-a, \lambda \\ c+\lambda, e+\lambda \end{matrix} \middle| 1 \right] \quad (4)$$

where  $\lambda = b + d - a - c - e$ . This transformation has further been employed by Milgram [14], where a systematic examination has been carried out for numerous  ${}_3F_2(1)$ -series identities.

Wimp [21] has shown that Watson’s formula cannot be generalized into the series  ${}_3F_2 \left[ \begin{matrix} a, b, c \\ d, 2c \end{matrix} \middle| 1 \right]$  with four free parameters  $\{a, b, c, d\}$  (see Zeilberger [22] for a shorter proof). Since then, attention has been turned to extending the series displayed in (1-3) by adjusting the parameters with specific integers  $m$  and  $n$ :

$${}_3F_2 \left[ \begin{matrix} a, b, c \\ 1+a-b+m, 1+a-c+n \end{matrix} \middle| 1 \right], {}_3F_2 \left[ \begin{matrix} a, b, c \\ \frac{1+a+b+m}{2}, 2c+n \end{matrix} \middle| 1 \right], {}_3F_2 \left[ \begin{matrix} a, 1-a+m, b \\ c, 1+2b-c+n \end{matrix} \middle| 1 \right].$$

When  $m = n = 0$ , they reduce to the original Dixon, Watson and Whipple series. By utilizing contiguous relations for the  ${}_3F_2$ -series (cf. Rainville [15, §48]), Lavoie *et al* [10–12] made an extensive investigation to these three series when  $m$  and  $n$  are small integers. Their experimental computation leads to collections of 25 formulae for generalized Watson series (the second one just displayed) in [10] and 38 formulae for generalized Dixon series (the first one just displayed) in [11], where the latter ones are further translated, via the Thomae transformation, into 38 formulae for generalized Whipple series (the third one just displayed) in [12]. For limiting relations, different proofs and other related works, the reader can find in the following papers [2–4, 9, 16–18].

Lewanowicz [13] made the first attempt to find analytical formula for the last three series with  $m$  and  $n$  being general integer parameters, who derived general expressions in terms of three known formulae (1–3). However for most of his formulae, the connection coefficients have to be determined implicitly by recurrence relations involving two variables, which are not convenient to practical evaluation.

The present paper aims at establishing analytical formulae explicitly for the generalized Watson–Whipple–Dixon series with two integer parameters  $m$  and  $n$ . In the next section, we shall prove four crucial preliminary lemmas that transform the generalized Watson series into Watson’s original one (2). This will be accomplished by combining the linearization method with the following summation theorem for terminating well–poised  ${}_5F_4$ -series due to Dougall [7] (cf. Bailey [1, §4.3]):

$$\left[ \begin{matrix} 1+a, 1+a-b-d \\ 1+a-b, 1+a-d \end{matrix} \right]_m = {}_5F_4 \left[ \begin{matrix} a, 1+\frac{a}{2}, b, d, -m \\ \frac{a}{2}, 1+a-b, 1+a-d, 1+a+m \end{matrix} \middle| 1 \right]. \quad (5)$$

According to the signs of  $m$  and  $n$ , four analytical formulae for the generalized Watson series with two extra integer parameters will be derived in Section 3. By means of the Thomae transformation (4), the generalized Whipple and Dixon series with two integer parameters  $m$  and  $n$  are converted to those for generalized Watson series in the last two sections 4 and 5.

Based on our analytical formulae, the corresponding commands in *Mathematica* are written to evaluate the series of three classes (Watson, Whipple and Dixon) for any specific integers  $m$  and  $n$ . We are limited to present twelve examples with four representatives for each class, even though it would be routine to produce longer list such as those given by Lavoie *et al* [10–12].

## 2. FOUR TRANSFORMATION LEMMAS

Define the two nonterminating  ${}_3F_2$ -series with four free parameters  $\{a, b, c, d\}$  by

$$U_m(a, b, c, d) = {}_3F_2 \left[ \begin{matrix} a, b, c \\ \frac{1+a+b+m}{2}, d \end{matrix} \middle| 1 \right], \quad (6)$$

$$V_n(a, b, c, d) = {}_3F_2 \left[ \begin{matrix} a, b, c \\ d, 2c+n \end{matrix} \middle| 1 \right]. \quad (7)$$

Then Watson’s series displayed in (2) results in the particular cases

$$U_0(a, b, c, 2c) = V_0(a, b, c, \frac{1+a+b}{2}).$$

By means of the linearization method, we shall show four transformations for the series  $U_m$  and  $V_n$  with respect to the signs of  $m$  and  $n$ . They will be utilized, in the next section, to derive explicit formulae for the generalized Watson series with two integer parameter  $m$  and  $n$ .

**§2.1. Transformation for  $U_m$ .** Observe that any  $m+1$  polynomials of degree  $m$  in  $k$  are linearly dependent. There exist  $m+1$  constants  $\{X_m^i\}$  such that

$$1 = \sum_{i=0}^m X_m^i (a+k)_i (b+k)_{m-i}. \quad (8)$$

Equating the coefficients of  $k^j$  with  $0 \leq j \leq m$ , we get from (8) a system of equations in  $\{X_m^i\}$ , which can be resolved by the following *Mathematica* commands for any (not too large) specific natural number  $m$ :

```
pp[x_, n_] := Pochhammer[x, n]
uu[k_, m_] := Sum[xx[i, m]*pp[a+k, i]*pp[b+k, m-i], {i, 0, m}]
ee[m_] := Table[Coefficient[uu[k, m], k, i] == If[i==0, 1, 0], {i, 0, m}]
ww[m_] := Flatten[MapAll[Factor, Solve[ee[m], Table[xx[i, m], {i, 0, m}]]]]
```

The first few outputs for “ww[m]” with  $1 \leq m \leq 5$  have nice pattern and suggest the following solution

$$X_m^i = (-1)^i \binom{m}{i} \frac{b-a+m-2i}{(b-a-i)_{m+1}} \quad \text{for } i = 0, 1, \dots, m. \quad (9)$$

This is equivalent to the following equality

$$\frac{(b-a)_m}{(b+k)_m} = {}_4F_3 \left[ \begin{matrix} a-b-m, 1 + \frac{a-b-m}{2}, a+k, -m \\ \frac{a-b-m}{2}, 1-b-k-m, 1+a-b \end{matrix} \middle| -1 \right] \quad (10)$$

which can be justified by the limiting case of Dougall’s theorem for  ${}_5F_4$ -series displayed in (5), specified by  $a \rightarrow a-b-m$ ,  $b \rightarrow a+k$  and  $d \rightarrow \infty$ .

Now considering the composite series and then interchanging the summation order, we have the double series expression

$$\begin{aligned}
U_m(a, b, c, d) &= \sum_{k \geq 0} \left[ 1, \frac{a, b, c}{1+a+b+m}, d \right]_k \sum_{i=0}^m X_m^i (a+k)_i (b+k)_{m-i} \\
&= \sum_{i=0}^m X_m^i \sum_{k \geq 0} \left[ 1, \frac{a, b, c}{1+a+b+m}, d \right]_k (a+k)_i (b+k)_{m-i} \\
&= \sum_{i=0}^m X_m^i (a)_i (b)_{m-i} \sum_{k \geq 0} \left[ \frac{a+i, b+m-i, c}{1, \frac{1+a+b+m}{2}}, d \right]_k
\end{aligned}$$

which gives rise to the following equation

$$U_m(a, b, c, d) = \sum_{i=0}^m X_m^i (a)_i (b)_{m-i} U_0(a+i, b+m-i, c, d). \quad (11)$$

Substituting the explicit expression of  $X_m^i$  into the last equation and then simplifying the result, we get the first transformation formula.

**Lemma 1** ( $m \in \mathbb{N}_0$ ).

$$\begin{aligned}
U_m(a, b, c, d) &= \frac{(b)_m}{(b-a)_m} \sum_{i=0}^m \binom{m}{i} U_0(a+i, b+m-i, c, d) \\
&\quad \times \frac{a-b-m+2i}{a-b-m} \left[ 1 + \frac{a, a-b-m}{1+a-b, 1-b-m} \right]_i.
\end{aligned}$$

The special case  $d = 2c$  of this lemma has been discovered by Lewanowicz [13, Equation 2.2]. To our knowledge, this is the only case that the explicit coefficients are previously determined for expressing the generalized Watson series with two integer parameters  $m$  and  $n$  in terms of Watson's original one (2).

**§2.2. Transformation for  $U_{-m}$ .** In this case, there exist  $m+1$  constants  $\{\mathcal{X}_m^i\}$  independent of  $k$  such that

$$\left( \frac{1+a+b-m}{2} + k \right)_m = \sum_{i=0}^m \mathcal{X}_m^i (a+k)_i (b+k)_{m-i}. \quad (12)$$

Analogously with the help of the *Mathematica* commands below

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uu[k_, m_] := pp[k+(1+a+b-m)/2, m] - Sum[xx[i, m] * pp[a+k, i] * pp[b+k, m-i], {i, 0, m}]
ee[m_] := Table[Coefficient[uu[k, m], k, i] == 0, {i, 0, m}]
ww[m_] := Flatten[MapAll[Factor, Solve[ee[m], Table[xx[i, m], {i, 0, m}]]]]

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the coefficients  $\mathcal{X}_m^i$  are detected (but to be verified) as follows:

$$\mathcal{X}_m^i = \left( \frac{1-a+b-m}{2} \right)_m \binom{m}{i} \frac{b-a+m-2i}{(b-a-i)_{m+1}} \quad \text{for } i = 0, 1, \dots, m. \quad (13)$$

This is equivalent to the following equality

$$\left[ \frac{b-a, \frac{1+a+b-m}{2} + k}{b+k, \frac{1-a+b-m}{2}} \right]_m = {}_4F_3 \left[ \frac{a-b-m, 1 + \frac{a-b-m}{2}, a+k, -m}{\frac{a-b-m}{2}, 1-b-k-m, 1+a-b} \middle| 1 \right] \quad (14)$$

which turns to be a special case of Dougall's formula (5) under the parameter specifications  $a \rightarrow a-b-m$ ,  $b \rightarrow a+k$  and  $d \rightarrow (1+a-b-m)/2$ .

Then we can manipulate the following composite series

$$\begin{aligned}
U_{-m}(a, b, c, d) &= \sum_{k \geq 0} \left[ 1, \frac{a, b, c}{1+a+b-m}, d \right]_k \sum_{i=0}^m \mathcal{X}_m^i \frac{(a+k)_i (b+k)_{m-i}}{\left(\frac{1+a+b-m}{2} + k\right)_m} \\
&= \sum_{i=0}^m \mathcal{X}_m^i \sum_{k \geq 0} \left[ 1, \frac{a, b, c}{1+a+b-m}, d \right]_k \frac{(a+k)_i (b+k)_{m-i}}{\left(\frac{1+a+b-m}{2} + k\right)_m} \\
&= \sum_{i=0}^m \mathcal{X}_m^i \frac{(a)_i (b)_{m-i}}{\left(\frac{1+a+b-m}{2}\right)_m} \sum_{k \geq 0} \left[ a+i, b+m-i, c \right]_k \left[ 1, \frac{1+a+b+m}{2}, d \right]_k
\end{aligned}$$

which results in the following equation

$$U_{-m}(a, b, c, d) = \sum_{i=0}^m \mathcal{X}_m^i \frac{(a)_i (b)_{m-i}}{\left(\frac{1+a+b-m}{2}\right)_m} U_0(a+i, b+m-i, c, d). \quad (15)$$

Substituting the explicit expression of  $\mathcal{X}_m^i$  into the last equation and then simplifying the result, we find the second transformation formula.

**Lemma 2** ( $m \in \mathbb{N}_0$ ).

$$\begin{aligned}
U_{-m}(a, b, c, d) &= \left[ b, \frac{1-a+b-m}{2} \right]_m \sum_{i=0}^m (-1)^i \binom{m}{i} U_0(a+i, b+m-i, c, d) \\
&\quad \times \frac{a-b-m+2i}{a-b-m} \left[ 1+a-b, 1-b-m \right]_i.
\end{aligned}$$

**§2.3. Transformation for  $V_n$ .** Now that any  $n+1$  polynomials of degree  $n$  in  $k$  are linearly dependent, there exist  $n+1$  constants  $\{Y_n^j\}$  such that

$$1 = \sum_{j=0}^n Y_n^j \langle k \rangle_j \langle 2c+n+k-1 \rangle_{n-j}. \quad (16)$$

Equating the coefficients of  $k^i$  with  $0 \leq i \leq n$ , the *Mathematica* commands below

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qq[x_, n_] := (-1)^n * Pochhammer[-x, n]
vv[k_, n_] := Sum[yy[j, n] * qq[k, j] * pp[2c+k+j, n-j], {j, 0, n}]
ee[n_] := Table[Coefficient[vv[k, n], k, j] == If[j == 0, 1, 0], {j, 0, n}]
ww[n_] := Flatten[MapAll[Factor, Solve[ee[n], Table[yy[j, n], {j, 0, n}]]]]

```

reveal the following experimental solution

$$Y_n^j = (-1)^j \binom{n}{j} \frac{2j-1+2c}{(j-1+2c)_{n+1}} \quad \text{for } j = 0, 1, \dots, n. \quad (17)$$

This is equivalent to the following equality

$$\frac{(2c)_n}{(2c+k)_n} = {}_4F_3 \left[ \begin{matrix} 2c-1, c+\frac{1}{2}, -k, -n \\ c-\frac{1}{2}, 2c+k, 2c+n \end{matrix} \middle| -1 \right] \quad (18)$$

which can be justified again by the limiting case of Dougall's sum (5), corresponding to the parameter settings  $a \rightarrow 2c-1$ ,  $b \rightarrow -k$  and  $d \rightarrow \infty$ .

Now considering the composite series and then interchanging the summation order, we have the double series expression

$$\begin{aligned}
V_n(a, b, c, d) &= \sum_{k \geq 0} \left[ \begin{matrix} a, b, c \\ 1, d, 2c+n \end{matrix} \right]_k \sum_{j=0}^n Y_n^j \langle k \rangle_j \langle 2c+n+k-1 \rangle_{n-j} \\
&= \sum_{j=0}^n Y_n^j \sum_{k \geq 0} \left[ \begin{matrix} a, b, c \\ 1, d, 2c+n \end{matrix} \right]_k \langle k \rangle_j \langle 2c+n+k-1 \rangle_{n-j} \\
&= \sum_{j=0}^n Y_n^j \frac{(a)_j (b)_j (c)_j}{(d)_j (2c+n)_{2j-n}} \sum_{k \geq 0} \left[ \begin{matrix} a+j, b+j, c+j \\ 1, d+j, 2c+2j \end{matrix} \right]_k
\end{aligned}$$

which results in the following equation

$$V_n(a, b, c, d) = \sum_{j=0}^n Y_n^j \frac{(a)_j (b)_j (c)_j}{(d)_j (2c+n)_{2j-n}} V_0(a+j, b+j, c+j, d+j). \quad (19)$$

Substituting the explicit expression of  $Y_n^j$  into the last equation and then simplifying the result, we have the third transformation formula.

**Lemma 3** ( $n \in \mathbb{N}_0$ ).

$$V_n(a, b, c, d) = \sum_{j=0}^n (-1)^j \binom{n}{j} \frac{(2c-1)_j}{(2c-1)_{2j}} \left[ \begin{matrix} a, b, c \\ d, 2c+n \end{matrix} \right]_j V_0(a+j, b+j, c+j, d+j).$$

§2.4. **Transformation for  $V_{-n}$ .** Analogously there exist  $n+1$  constants  $\{\mathcal{Y}_n^j\}$  independent of  $k$  such that

$$(c-n+k)_n = \sum_{j=0}^n \mathcal{Y}_n^j \langle k \rangle_j \langle 2c-n+k-1 \rangle_{n-j}. \quad (20)$$

By utilizing the *Mathematica* commands below

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vv[k_, n_] := pp[c-n+k, n] - Sum[yy[j, n] * qq[k, j] * pp[2c-2n+k+j, n-j], {j, 0, n}]
ee[n_] := Table[Coefficient[vv[k, n], k, j] == 0, {j, 0, n}]
ww[n_] := Flatten[MapAll[Factor, Solve[ee[n], Table[yy[j, n], {j, 0, n}]]]]

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we can similarly figure out the coefficients  $\mathcal{Y}_n^j$  as follows:

$$\mathcal{Y}_n^j = \binom{n}{j} \frac{2c+2j-2n-1}{2c+j-n-1} \frac{(1-c)_n}{(2+n-j-2c)_n} \quad \text{for } j = 0, 1, \dots, n. \quad (21)$$

This is equivalent to the following equality

$$\left[ \begin{matrix} c-n+k, 2c-2n \\ c-n, 2c-2n+k \end{matrix} \right]_n = {}_4F_3 \left[ \begin{matrix} 2c-2n-1, c-n+\frac{1}{2}, -k, -n \\ c-n-\frac{1}{2}, 2c-2n+k, 2c-n \end{matrix} \middle| 1 \right] \quad (22)$$

which is, in fact, a particular case of Dougall's theorem (5) with the parameters being specified by  $a \rightarrow 2c-2n-1$ ,  $b \rightarrow -k$  and  $d \rightarrow c-n$ .

For the following composite series, it can be reformulated as

$$\begin{aligned} V_{-n}(a, b, c, d) &= \sum_{k \geq 0} \left[ \begin{matrix} a, b, c \\ 1, d, 2c - n \end{matrix} \right]_k \sum_{j=0}^n \mathfrak{y}_n^j \frac{\langle k \rangle_j \langle 2c - n + k - 1 \rangle_{n-j}}{(c - n + k)_n} \\ &= \sum_{j=0}^n \mathfrak{y}_n^j \sum_{k \geq 0} \left[ \begin{matrix} a, b, c \\ 1, d, 2c - n \end{matrix} \right]_k \frac{\langle k \rangle_j \langle 2c - n + k - 1 \rangle_{n-j}}{(c - n + k)_n} \\ &= \sum_{j=0}^n \mathfrak{y}_n^j \frac{(a)_j (b)_j (c)_{j-n}}{(d)_j (2c - n)_{2j-n}} \sum_{k \geq 0} \left[ \begin{matrix} a + j, b + j, c + j - n \\ 1, d + j, 2c + 2j - 2n \end{matrix} \right]_k \end{aligned}$$

which leads to the following equation

$$V_{-n}(a, b, c, d) = \sum_{j=0}^n \mathfrak{y}_n^j \frac{(a)_j (b)_j (c)_{j-n}}{(d)_j (2c - n)_{2j-n}} V_0(a + j, b + j, c + j - n, d + j). \quad (23)$$

Substituting the explicit expression of  $\mathfrak{y}_n^j$  into the last equation and then simplifying the result, we obtain the fourth transformation formula.

**Lemma 4** ( $n \in \mathbb{N}_0$ ).

$$V_{-n}(a, b, c, d) = \sum_{j=0}^n \binom{n}{j} \frac{(2c - 2n - 1)_j}{(2c - 2n - 1)_{2j}} \left[ \begin{matrix} a, b, c - n \\ d, 2c - n \end{matrix} \right]_j V_0(a + j, b + j, c + j - n, d + j).$$

### 3. EXTENSION OF WATSON'S ${}_3F_2$ -SERIES

For the two integer parameters  $m$  and  $n$ , the extended Watson series is defined by

$$W_{m,n}(a, b, c) := {}_3F_2 \left[ \begin{matrix} a, b, c \\ \frac{1+a+b+m}{2}, 2c+n \end{matrix} \middle| 1 \right] \quad (24)$$

which reduces to Watson's original one displayed in (2) when  $m = n = 0$ . According to Lemmas 1 and 2, this series

$$W_{m,n}(a, b, c) = U_m(a, b, c, 2c + n) \quad (25)$$

can be written as linear sum of

$$U_0(a', b', c', 2c' + n) = V_n(a', b', c', \frac{1+a'+b'}{2}) \quad (26)$$

which can further be expressed, in view of Lemmas 3 and 4, in terms of Watson's original series

$$W_{0,0}(a'', b'', c'') = V_0(a'', b'', c'', \frac{1+a''+b''}{2}). \quad (27)$$

This leads to the double sum expression of  $W_{m,n}(a, b, c)$  in  $(|m|+1)(|n|+1)$  terms of  $W_{0,0}(a'', b'', c'')$ , which permits us to evaluate  $W_{m,n}(a, b, c)$  for any specific integers  $m$  and  $n$ . The results are displayed in the following four theorems with respect to the signs of  $m$  and  $n$ .

For  $m \geq 0$  and  $n \geq 0$ , combining Lemma 1 with Lemma 3 yields the first summation theorem, which expresses  $W_{m,n}$  explicitly in terms of  $W_{0,0}$ .

**Theorem 5** ( $m, n \in \mathbb{N}_0$ ).

$$\begin{aligned} W_{m,n}(a, b, c) &= \sum_{i=0}^m \sum_{j=0}^n (-1)^{i+j} \binom{m}{i} \binom{n}{j} \frac{a - b - m + 2i}{a - b - m + i} \frac{(a)_{i+j} (b)_{m-i+j}}{(b - a - i)_m (2c - 1)_{2j}} \\ &\quad \times \left[ \begin{matrix} c, 2c - 1 \\ \frac{1+a+b+m}{2}, 2c + n \end{matrix} \right]_j W_{0,0}(a + i + j, b + m - i + j, c + j). \end{aligned}$$

For  $m \geq 0$  and  $n \leq 0$ , combining Lemma 1 with Lemma 4 gives the second summation theorem, which expresses  $W_{m,n}$  explicitly in terms of  $W_{0,0}$ .

**Theorem 6** ( $m, n \in \mathbb{N}_0$ ).

$$W_{m,-n}(a, b, c) = \sum_{i=0}^m \sum_{j=0}^n (-1)^i \binom{m}{i} \binom{n}{j} \frac{a-b-m+2i}{a-b-m+i} \frac{(a)_{i+j}(b)_{m-i+j}}{(b-a-i)_m (2c-2n-1)_{2j}} \\ \times \left[ \frac{c-n, 2c-2n-1}{\frac{1+a+b+m}{2}, 2c-n} \right]_j W_{0,0}(a+i+j, b+m-i+j, c+j-n).$$

For  $m \leq 0$  and  $n \geq 0$ , combining Lemma 2 with Lemma 3 results in the third summation theorem, which expresses  $W_{m,n}$  explicitly in terms of  $W_{0,0}$ .

**Theorem 7** ( $m, n \in \mathbb{N}_0$ ).

$$W_{-m,n}(a, b, c) = \frac{\left(\frac{1-a+b-m}{2}\right)_m}{\left(\frac{1+a+b-m}{2}\right)_m} \sum_{i=0}^m \sum_{j=0}^n (-1)^j \binom{m}{i} \binom{n}{j} \frac{a-b-m+2i}{a-b-m+i} \frac{(b)_{m-i+j}}{(b-a-i)_m} \\ \times \frac{(a)_{i+j}}{(2c-1)_{2j}} \left[ \frac{c, 2c-1}{\frac{1+a+b+m}{2}, 2c+n} \right]_j W_{0,0}(a+i+j, b+m-i+j, c+j).$$

For  $m \leq 0$  and  $n \leq 0$ , combining Lemma 2 with Lemma 4 leads to the fourth summation theorem, which expresses  $W_{m,n}$  explicitly in terms of  $W_{0,0}$ .

**Theorem 8** ( $m, n \in \mathbb{N}_0$ ).

$$W_{-m,-n}(a, b, c) = \frac{\left(\frac{1-a+b-m}{2}\right)_m}{\left(\frac{1+a+b-m}{2}\right)_m} \sum_{i=0}^m \sum_{j=0}^n \binom{m}{i} \binom{n}{j} \frac{a-b-m+2i}{a-b-m+i} \left[ \frac{c-n, 2c-2n-1}{\frac{1+a+b+m}{2}, 2c-n} \right]_j \\ \times \frac{(a)_{i+j}(b)_{m-i+j}}{(b-a-i)_m (2c-2n-1)_{2j}} W_{0,0}(a+i+j, b+m-i+j, c+j-n).$$

These four analytical formulae constitute a complete scheme to evaluate the generalized Watson series  $W_{m,n}$  for any two specific integers  $m$  and  $n$ . Four formulae are presented here as exemplification.

**Example 9.**

$$W_{1,1}(a, b, c) = \frac{2^{4c-2} \Gamma(1 + \frac{a+b}{2}) \Gamma(c + \frac{1}{2}) \Gamma(c - \frac{a+b}{2}) / (a-b)}{\pi^{3/2} \Gamma(a) \Gamma(b) \Gamma(1+2c-a) \Gamma(1+2c-b)} \\ \times \left[ (2c-a+b) \Gamma\left(\frac{1+a}{2}\right) \Gamma\left(\frac{b}{2}\right) \Gamma\left(\frac{1}{2} + c - \frac{a}{2}\right) \Gamma\left(1 + c - \frac{b}{2}\right) \right. \\ \left. - (2c+a-b) \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{1+b}{2}\right) \Gamma\left(1 + c - \frac{a}{2}\right) \Gamma\left(\frac{1}{2} + c - \frac{b}{2}\right) \right].$$

**Example 10.**

$$W_{1,-1}(a, b, c) = \frac{2^{4c-5} \Gamma(1 + \frac{a+b}{2}) \Gamma(c - \frac{1}{2}) \Gamma(c - \frac{a+b}{2}) / (a-b)}{\pi^{3/2} \Gamma(a) \Gamma(b) \Gamma(2c-a-1) \Gamma(2c-b-1)} \\ \times \left[ \Gamma\left(\frac{1+a}{2}\right) \Gamma\left(\frac{b}{2}\right) \Gamma\left(c - \frac{1+a}{2}\right) \Gamma\left(c - \frac{b}{2}\right) \right. \\ \left. - \Gamma\left(\frac{a}{2}\right) \Gamma\left(\frac{1+b}{2}\right) \Gamma\left(c - \frac{a}{2}\right) \Gamma\left(c - \frac{1+b}{2}\right) \right].$$



**Example 11.**

$$\begin{aligned} W_{-1,1}(a, b, c) &= \frac{2^{4c-2}\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(c+\frac{1}{2}\right)\Gamma\left(1+c-\frac{a+b}{2}\right)}{\pi^{3/2}\Gamma(a)\Gamma(b)\Gamma(1+2c-a)\Gamma(1+2c-b)} \\ &\times \left[ \Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{1}{2}+c-\frac{a}{2}\right)\Gamma\left(1+c-\frac{b}{2}\right) \right. \\ &\quad \left. + \Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(1+c-\frac{a}{2}\right)\Gamma\left(\frac{1}{2}+c-\frac{b}{2}\right) \right]. \end{aligned}$$

**Example 12.**

$$\begin{aligned} W_{-1,-1}(a, b, c) &= \frac{2^{4c-7}\Gamma\left(\frac{a+b}{2}\right)\Gamma\left(c-\frac{1}{2}\right)\Gamma\left(c-1-\frac{a+b}{2}\right)}{\pi^{3/2}\Gamma(a)\Gamma(b)\Gamma(2c-a-1)\Gamma(2c-b-1)} \\ &\times \left[ (2c-a+b-2)\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(c-\frac{1+a}{2}\right)\Gamma\left(c-\frac{b}{2}\right) \right. \\ &\quad \left. + (2c+a-b-2)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+b}{2}\right)\Gamma\left(c-\frac{a}{2}\right)\Gamma\left(c-\frac{1+b}{2}\right) \right]. \end{aligned}$$

#### 4. EXTENSION OF WHIPPLE'S ${}_3F_2$ -SERIES

For the two integer parameters  $m$  and  $n$ , define the extended Whipple series by

$$\Omega_{m,n}(a, b, c) = {}_3F_2 \left[ \begin{matrix} a, 1-a+m, b \\ c, 1+2b-c+n \end{matrix} \middle| 1 \right] \quad (28)$$

which reduces to Whipple's original one displayed in (3) when  $m = n = 0$ . Applying the Thomae transformation (4), we have

$$\begin{aligned} \Omega_{m,n}(a, b, c) &= \Gamma \left[ \begin{matrix} c, b-m+n, 1+2b-c+n \\ a, 1-a+b+n, 2b-m+n \end{matrix} \right] \\ &\times {}_3F_2 \left[ \begin{matrix} c-a, 1+2b-a-c+n, b-m+n \\ 1-a+b+n, 2b-m+n \end{matrix} \middle| 1 \right]. \end{aligned} \quad (29)$$

Identifying the last  ${}_3F_2$ -series with  $W_{n,m-n}$ , we get the following relation:

$$\begin{aligned} \Omega_{m,n}(a, b, c) &= \Gamma \left[ \begin{matrix} c, b-m+n, 1+2b-c+n \\ a, 1-a+b+n, 2b-m+n \end{matrix} \right] \\ &\times W_{n,m-n}(c-a, 1+2b-a-c+n, b-m+n). \end{aligned} \quad (30)$$

This transformation formula converts the generalized Whipple series into the generalized Watson series. Combining it with Theorems 5–8, we can evaluate the generalized Whipple series  $\Omega_{m,n}(a, b, c)$  for any specific integers  $m$  and  $n$ . Four examples are displayed below.

**Example 13.**

$$\begin{aligned} \Omega_{1,1}(a, b, c) &= \frac{2^{2b-3}\Gamma(c)\Gamma(2b-c+2)/\{(1-a)(1+b-c)\}}{\pi\Gamma(a+c-2)\Gamma(c-a)\Gamma(a+2b-c)\Gamma(2+2b-a-c)} \\ &\times \left[ \Gamma\left(\frac{a+c-2}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)\Gamma\left(b+\frac{1+a-c}{2}\right)\Gamma\left(b+\frac{2-a-c}{2}\right) \right. \\ &\quad \left. - \Gamma\left(\frac{a+c-1}{2}\right)\Gamma\left(\frac{c-a}{2}\right)\Gamma\left(b+\frac{a-c}{2}\right)\Gamma\left(b+\frac{3-a-c}{2}\right) \right]. \end{aligned}$$

**Example 14.**

$$\begin{aligned}\Omega_{-1,-1}(a, b, c) &= \frac{2^{2b-3}\Gamma(c)\Gamma(2b-c)}{\pi\Gamma(a+c)\Gamma(c-a)\Gamma(a+2b-c)\Gamma(2b-a-c)} \\ &\times \left[ \Gamma\left(\frac{c-a}{2}\right)\Gamma\left(\frac{1+a+c}{2}\right)\Gamma\left(b+\frac{1-a-c}{2}\right)\Gamma\left(b+\frac{a-c}{2}\right) \right. \\ &\quad \left. + \Gamma\left(\frac{a+c}{2}\right)\Gamma\left(\frac{1-a+c}{2}\right)\Gamma\left(b+\frac{1+a-c}{2}\right)\Gamma\left(b-\frac{a+c}{2}\right) \right].\end{aligned}$$

**Example 15.**

$$\begin{aligned}\Omega_{2,1}(a, b, c) &= \frac{2^{2b-4}\Gamma(c)\Gamma(2b-c+2)/\{(a-1)(a-2)(b-1)(b-c+1)\}}{\pi\Gamma(a+c-3)\Gamma(c-a)\Gamma(a+2b-c-1)\Gamma(2b-a-c+2)} \\ &\times \left[ (c-2b)\Gamma\left(\frac{a+c-2}{2}\right)\Gamma\left(\frac{1-a+c}{2}\right)\Gamma\left(b+\frac{a-c-1}{2}\right)\Gamma\left(b+\frac{2-a-c}{2}\right) \right. \\ &\quad \left. + (c-2)\Gamma\left(\frac{a+c-3}{2}\right)\Gamma\left(\frac{c-a}{2}\right)\Gamma\left(b+\frac{a-c}{2}\right)\Gamma\left(b+\frac{3-a-c}{2}\right) \right].\end{aligned}$$

**Example 16.**

$$\begin{aligned}\Omega_{-2,-1}(a, b, c) &= \frac{2^{2b-2}\Gamma(c)\Gamma(2b-c)}{\pi\Gamma(a+c+1)\Gamma(c-a)\Gamma(a+2b-c+1)\Gamma(2b-a-c)} \\ &\times \left[ (2b-c)\Gamma\left(\frac{2+a+c}{2}\right)\Gamma\left(\frac{1+c-a}{2}\right)\Gamma\left(b+\frac{1+a-c}{2}\right)\Gamma\left(b-\frac{a+c}{2}\right) \right. \\ &\quad \left. + c\Gamma\left(\frac{1+a+c}{2}\right)\Gamma\left(\frac{c-a}{2}\right)\Gamma\left(b+\frac{2+a-c}{2}\right)\Gamma\left(b+\frac{1-a-c}{2}\right) \right].\end{aligned}$$

## 5. ALMOST-POISED ${}_3F_2$ -SERIES

For the two integer parameters  $m$  and  $n$ , define the almost-poised  ${}_3F_2$ -series by

$$\mathcal{D}_{m,n}(a, b, c) = {}_3F_2 \left[ \begin{matrix} a, & b, & c \\ 1+a-b+m, & 1+a-c+n \end{matrix} \middle| 1 \right] \quad (31)$$

which reduces to Dixon's original one displayed in (1) when  $m = n = 0$ . Applying the Thomae transformation (4), we get

$$\begin{aligned}\mathcal{D}_{m,n}(a, b, c) &= \Gamma \left[ \begin{matrix} 1+a-b+m, 1+a-c+n, 2+a-2b-2c+m+n \\ b, 2+a-2b-c+m+n, 2+2a-2b-2c+m+n \end{matrix} \right] \\ &\times {}_3F_2 \left[ \begin{matrix} 1+a-2b+m, 2+a-2b-2c+m+n, 1+a-b-c+n \\ 2+a-2b-c+m+n, 2+2a-2b-2c+m+n \end{matrix} \middle| 1 \right].\end{aligned} \quad (32)$$

Identifying the last  ${}_3F_2$ -series with  $W_{n,m-n}$ , we obtain the following relation:

$$\begin{aligned}\mathcal{D}_{m,n}(a, b, c) &= \Gamma \left[ \begin{matrix} 1+a-b+m, 1+a-c+n, 2+a-2b-2c+m+n \\ b, 2+a-2b-c+m+n, 2+2a-2b-2c+m+n \end{matrix} \right] \\ &\times W_{n,m-n}(1+a-2b+m, 2+a-2b-2c+m+n, 1+a-b-c+n).\end{aligned} \quad (33)$$

This transformation formula converts the almost-poised  ${}_3F_2$ -series into the generalized Watson series again. Combining it with Theorems 5–8, we can evaluate the almost-poised  ${}_3F_2$ -series  $\mathcal{D}_{m,n}(a, b, c)$  for any specific integers  $m$  and  $n$ . Four formulae are exhibited as examples.

**Example 17.**

$$\begin{aligned} \mathcal{D}_{1,1}(a, b, c) &= \frac{2^{1+2a-2b-2c}\Gamma(a-b+2)\Gamma(a-c+2)/\{(b-1)(1-c)\}}{\pi\Gamma(a)\Gamma(a-2b+2)\Gamma(a-2c+2)\Gamma(a-b-c+2)} \\ &\times \left[ \Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{2+a}{2}-b\right)\Gamma\left(\frac{2+a}{2}-c\right)\Gamma\left(\frac{5+a}{2}-b-c\right) \right. \\ &\quad \left. - \Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{3+a}{2}-b\right)\Gamma\left(\frac{3+a}{2}-c\right)\Gamma\left(\frac{4+a}{2}-b-c\right) \right]. \end{aligned}$$

**Example 18.**

$$\begin{aligned} \mathcal{D}_{-1,-1}(a, b, c) &= \frac{2^{2a-2b-2c-3}\Gamma(a-b)\Gamma(a-c)}{\pi\Gamma(a)\Gamma(a-2b)\Gamma(a-2c)\Gamma(a-b-c)} \\ &\times \left[ \Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{1+a}{2}-b\right)\Gamma\left(\frac{1+a}{2}-c\right)\Gamma\left(\frac{a}{2}-b-c\right) \right. \\ &\quad \left. + \Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b\right)\Gamma\left(\frac{a}{2}-c\right)\Gamma\left(\frac{1+a}{2}-b-c\right) \right]. \end{aligned}$$

**Example 19.**

$$\begin{aligned} \mathcal{D}_{2,2}(a, b, c) &= \frac{4^{b+c-a-3}\pi\Gamma(a-b+3)\Gamma(a-c+3)\Gamma(a-2b-2c+6)}{(b-1)(b-2)(c-2)(c-1)\Gamma(a-b-c+3)} \\ &\times \left[ \frac{a^2-2ab-2ac+2bc+5a-2b-2c+2}{\Gamma\left(\frac{1+a}{2}\right)\Gamma\left(\frac{4+a}{2}-b\right)\Gamma\left(\frac{4+a}{2}-c\right)\Gamma\left(\frac{7+a}{2}-b-c\right)} \right. \\ &\quad \left. - \frac{4}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{3+a}{2}-b\right)\Gamma\left(\frac{3+a}{2}-c\right)\Gamma\left(\frac{6+a}{2}-b-c\right)} \right]. \end{aligned}$$

**Example 20.**

$$\begin{aligned} \mathcal{D}_{-2,-2}(a, b, c) &= \frac{4^{1+b+c-a}\pi\Gamma(a-b-1)\Gamma(a-c-1)\Gamma(a-2b-2c-2)}{\Gamma(a-b-c-1)} \\ &\times \left[ \frac{a^2-2ab-2ac+2bc-3a+2b+2c+2}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{a}{2}-b\right)\Gamma\left(\frac{a}{2}-c\right)\Gamma\left(\frac{a-1}{2}-b-c\right)} \right. \\ &\quad \left. + \frac{4}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{a-1}{2}-b\right)\Gamma\left(\frac{a-1}{2}-c\right)\Gamma\left(\frac{a-2}{2}-b-c\right)} \right]. \end{aligned}$$

Before concluding the paper, we would like to point out that there exists a common generalization of bilateral series due to M. Jackson [8] for both Watson's theorem (2) and Whipple's theorem (3). The interested reader is encouraged to pursue the corresponding bilateral counterparts for the formulae displayed in this paper.

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