Root polytope and partitions

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Let Φ be a crystallographic reduced root system in the euclidean space E with scalar product $(\cdot, \cdot)$. Let R be the $\mathbb{Z}$–span of Φ in E and define $|\gamma|$, the *length* of γ, as the minimum $r \geq 0$ such that there exist r roots $\beta_1, \beta_2, \ldots, \beta_r$ with $\gamma = \beta_1 + \beta_2 + \cdots + \beta_r$. So $|\gamma|$ is the size of a minimal partition of γ in roots.

Our aim is to describe the map $R \ni \gamma \longmapsto |\gamma| \in \mathbb{N}$. (It is called the word length with respect to Φ in [1], see also [12], and [14]). As we show it is related to the convex hull $\mathcal{P}_\Phi$ of Φ in E, called the *root polytope*. Given an element $\lambda \in E$, let $H(\lambda)$ be the closed half–space of E defined by $\{u \in E \,|\, (\lambda, u) \leq 1\}$. If F is a facet of $\mathcal{P}_\Phi$ let $\lambda_F \in E$ be such that $(\lambda_F, u) = 1$ if $u \in F$ and $(\lambda_F, u) < 1$ for $u \in \mathcal{P}_\Phi \setminus F$; then $\mathcal{P}_\Phi = \cap_F H(\lambda_F)$ is a half–space presentation of $\mathcal{P}_\Phi$. Moreover let $V(F) \doteq F \cap \Phi$ be the set of roots in F and $C(V(F))$ be the $\mathbb{Q}^+$–cone over $V(F)$, i.e. the set of non–negative rational linear combinations of elements of $V(F)$. Our main result is the following formula.

Theorem A. *For any $\gamma \in R$ we have*

$$|\gamma| = \max_F \lceil (\lambda_F, \gamma) \rceil.$$

So if $\gamma \in C(V(F))$ for some facet F then $|\gamma| = \lceil (\lambda_F, \gamma) \rceil$.

Hence we see that the length map is quasi–linear, i.e. linear up to taking the integral part, on the cones over the facets of $\mathcal{P}_\Phi$.

Our proof of this theorem is straigthforward but has a unique interesting point; the map $\gamma \longmapsto |\gamma|$ needs to be evaluated at the generators of the monoid $M(F) \doteq C(V(F)) \cap R$, with F a face of $\mathcal{P}_\Phi$. Let $N(F)$ be the $\mathbb{N}$–span of $V(F)$ and $Z(F)$ be the $\mathbb{Z}$–span of $V(F)$; $N(F)$ is a submonoid of $M(F)$ and we say that the face F is *normal* if $C(V(F)) \cap Z(F) = N(F)$. Notice, however, that it is to be expected that $M(F)$ is larger than $N(F)$ since for some faces $Z(F)$ is a proper sublattice of R. We call $M(F)$ the *integral closure* of $N(F)$ in R and we say that F is *integrally closed* in R if $M(F) = N(F)$. The relation of these monoids to certain toric varieties gives reason to these definitions; in particular the normality of a face F is equivalent to the normality of the toric variety whose coordinate ring is $\mathbb{C}[t^\beta \,|\, \beta \in V(F)]$. Similar varieties have been extensively studied over the years; see for example [11], [7] and the other papers cited there.

As stated above, we need to find the generators of $M(F)$; this computation uses the uniform description of $\mathcal{P}_\Phi$ given by Cellini and Marietti in [4] (see also Vinberg's paper [13] and [6], [7] for a generalization). Assuming here that Φ is irreducible, the faces of $\mathcal{P}_\Phi$ may be naturally defined in terms of the affine root system associated to Φ. Let us say that a simple root α is *maximal* if its complement in the affine Dynkin diagram of Φ is connected. Let $\check{\omega}_\alpha$ be the coweight dual to α and let θ be the highest root of Φ; then the *standard parabolic facet* $F(\alpha)$, with α a (simple) maximal root, is the set of elements $u \in E$ such that $\check{\omega}_\alpha(u) = \check{\omega}_\alpha(\theta)$. All facets of $\mathcal{P}_\Phi$ are conjugate of the standard parabolic facets by the Weyl group. Notice that this gives in particular an explicit half–space presentation of $\mathcal{P}_\Phi$ and allows for an effective computation of the length as in the above Theorem.

We are now ready to report our computation about the generators of $M(F)$. We prove that the intersection of $M(F)$ with a face of the cone $C(V(F))$ is the monoid $M(F')$ for some facet F' of a subsystem of Φ; hence we consider only the *proper* generators of $M(F)$, i.e. those not in the border of $C(V(F))$.

Theorem B. *Let F be a facet of $\mathcal{P}_\Phi$, then the proper generators for the monoid $M(F)$ not in $V(F)$ are as follows:*

- $2\omega_3$ *for the facet* $F(\alpha_3)$ *of type* B_3*,*
- $2\omega_2$ *for the facet* $F(\alpha_2)$ *of type* E_7*,*
- ω_2 *and* $2\omega_2$ *for the facet* $F(\alpha_2)$ *of type* E_8 *and*

- ω_1 *and* $2\omega_1$ *for the facet* $F(\alpha_1)$ *of type* G_2.

All other facets of any other type have no proper generator.

We develop a general theory for proper generators to a certain extend; but the proof of the above Theorem needs some simple computations and checks that are carried out on a case–by–case basis. As a consequence we have the following two results.

Corollary C. *Any face of the root polytope* $\mathcal{P}_\Phi$ *is normal.*

Corollary D. *The facet* $F(\alpha)$, $\alpha \in \Delta$ *a maximal root, is integrally closed in* R *if and only if* $\check{\omega}_\alpha(\theta) = 1$.

These integral closure properties may also be proved by finding a unimodular triangulation of the facets of the root polytope; see for example [1] where such a triangulation is given for type A, C and D via explicit realizations of these root systems. Similar polytopes related to root systems and their normality are studied via unimodular triangulation in [3], [8] and [9] while in [10] a combinatorial characterization of diagonally split toric varieties is used. Notice however, that, to our best knowledge, it is not known whether all root polytopes admit an unimodular triangulation suitable to prove the above corollaries, nor we know how to costruct a triangulation in a uniform way with respect to the root system type.

It is natural to consider also another type of length map. Let R^+ be the $\mathbb{N}$–span of the positive roots and to an element γ of R^+ let us associate the minimum number $|\gamma|_+$ of positive roots needed to write γ as sum of positive roots. So $|\gamma|_+$ is the size of a minimal partition of γ in positive roots. We call $|\gamma|_+$ the *positive length* of γ. It is clear that $|\gamma| \leq |\gamma|_+$ but in general this inequality is strict. Consider, for example, B_3 and the root $\beta \doteq \alpha_1 + \alpha_2 + 2\alpha_3$; we have $\gamma \doteq \beta - \alpha_2 = \alpha_1 + 2\alpha_3$ and this shows that $|\gamma| = 2$ while $|\gamma|_+ = 3$ since any root has connected support.

Corollary E. *The positive length map coincides with the length map only for the types* A_ℓ *and* C_ℓ.

For type A_ℓ we prove this theorem by comparing a direct formula for the positive lenght with the formula in Theorem A. For C_ℓ we use a different strategy exploiting a triangulation of the root polytope described in [5].

The different behavior of types A_ℓ and C_ℓ with respect to the length map is reflected in the following compatibility condition. Let us denote by $\mathcal{P}_\Phi^+$ the convex hull of the set $\Phi^+ \cup \{0\}$, called the *positive root polytope*, and let also $C(\Phi^+)$ be the non–negative rational cone generated by the positive roots. We say that Φ^+ is *polyhedral* if $C(\Phi^+)$ is a union of cones generated by subsets of roots of the faces of $\mathcal{P}_\Phi$. Equivalently, Φ^+ is polyhedral if $\mathcal{P}_\Phi^+ = \mathcal{P}_\Phi \cap C(\Phi^+)$.

Notice that for A_3 and C_3, Φ^+ is polyhedral while it is not for B_3 as one may see in Figure 1. Moreveor it is clear that Φ^+ is not polyhedral for G_2 (see for example the tables in [2]) and it is easy to show that Φ^+ is not polyhedral for D_4; further if Φ^+ is polyhedral then also the set of positive roots of a subsystem is polyhedral. Hence only A_ℓ and C_ℓ may have a polyhedral positive root set and, indeed, this is proved in [5].

So Φ^+ is polyhedral if and only if the positive lenght map coincides with the lenght map. This suggest that some result similar to the formula in Theorem A should holds also for $\gamma \longmapsto |\gamma|_+$ using a half–space presentation of $\mathcal{P}_\Phi^+$.

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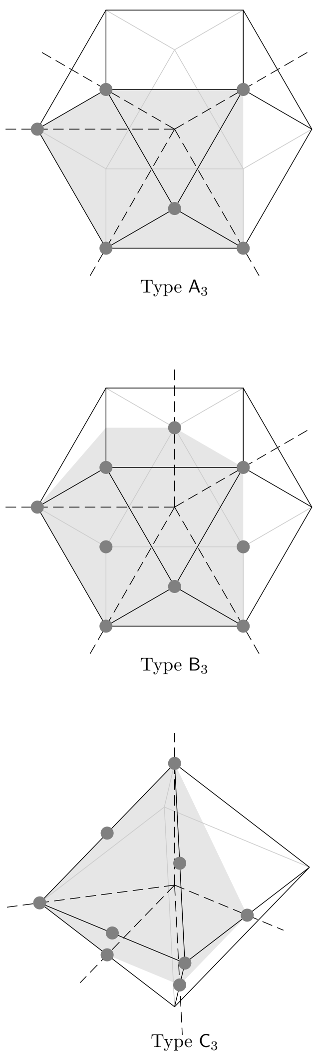

Figure 1. The rank 3 root polytopes

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