Plücker relations and spherical varieties: application to model varieties

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A standard monomial theory for an algebra A over a field $\Bbbk$ is given by a set of generators $\mathbb{A}$, together with a notion of standardness for monomials in $\mathbb{A}$ such that A is spanned by standard monomials as a $\Bbbk$–vector space; further the relations in A writing non standard monomials in terms of standard ones, called straightening relations, are "upper triangular". One of the main purpose of standard monomial theory is to replace the knoweldge of the equations defining a variety by the order requirement in the straightening relations. Indeed in many situation this weaker property is enough to prove geometric results like normality, Cohen-Maculay property, degeneration results and others. Moreover the order structure of the straightening relations allows to prove that they are generators for the ideal defining the algebra A as a quotient of the symmetric algebra $\mathsf{S}(\mathbb{A})$. The first example of such a theory dates back to Hodge study in [8] of the grassmannian of k–spaces in a n–dimensional vector space.

A standard monomial theory for flag and Schubert varieties has been developed over the years by Lakshmibai, Musili and Seshadri [9], this program culminated in the work of Littelmann [10] (see also [3]) where such a theory is defined in the generality of symmetrizable Kac–Moody groups.

At the same time, in [5] a standard monomial theory for the coordinate ring of SL_n was reduced to that of the grassmannian of n–spaces in a $2n$–dimensional space. Next this result was generalized in various directions by many authors (see the introduction in [4] for further details). In our paper [4], we shown how a standard monomial theory for certain classes of symmetric varieties may be described in terms of the Plücker relations of a suitable, maybe infinite dimensional, grassmannians. Moreover all previous known cases of this type of reduction are particular instances of our construction for symmetric varieties.

The first purpose of the present research is the development of a general framework for this reduction from the coordinate ring of a variety to the coordinate ring of a grassmannian. We propose how a suitable grassmannian for such process may be defined if we start with a spherical variety. However this proposal does not work in general for all spherical varieties, indeed various technical hypothesis must be met. It is however quite general and the hypothesis are fulfilled in many interesting cases. Let us explains our approach in more details.

Let G be a semisimple and connected algebraic group and let H be an algebraic subgroup such that $X \doteq G/H$ is spherical. Let Λ be the weight lattice and Λ^+ the monoid of dominant weights with respect to a fixed maximal torus and Borel subgroup of G. Denote by Ω^+ the monoid of spherical weights, i.e. of dominant weights λ such that the H–invariant subspace V_λ^H of the G–irreducible module V_λ is non–zero. Our first hypothesis is that Ω^+ is a free monoid; let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ be its generators. Then the coordinate ring A of X is generated by $V^*_{\varepsilon_1}, V^*_{\varepsilon_2}, \ldots, V^*_{\varepsilon_\ell}$. Our aim is to construct a standard monomial theory having as generators a basis of $V^*_{\varepsilon_1} \oplus V^*_{\varepsilon_2} \oplus \cdots \oplus V^*_{\varepsilon_\ell}$.

The main request is the existence of a Kac–Moody group K such that G is the semisimple part of a Levi of K with the following properties. There exists a suitable grassmannian $\mathcal{F}$ for K, a G–invariant Richardson variety $\mathcal{R} \subset \mathcal{F}$ and a line bundle $\mathcal{L}$ on $\mathcal{F}$ such that X may be embedded in a completion of $\mathcal{F}$, $H^0(\mathcal{R}, \mathcal{L}) \simeq V^*_{\varepsilon_1} \oplus V^*_{\varepsilon_2} \oplus \cdots \oplus V^*_{\varepsilon_\ell}$ and $\oplus_{n \geqslant 0} H^0(\mathcal{R}, \mathcal{L}^n)$ is isomorphic to the coordinate ring of G/H as G–modules. In a way, this group K is a bigger group of "hidden" symmetries for X.

Further we require the existence of an additive map $\mathsf{gr} : \Omega^+ \longrightarrow \mathbb{N}$ such that the following compatibility with tensor product of spherical modules holds: for all $\mu, \lambda, \nu \in \Omega^+$ such that $V^*_\nu \subset V^*_\lambda \otimes V^*_\mu$ we have $\mathsf{gr}(\nu) \leqslant \mathsf{gr}(\lambda + \mu)$. We require also that the generators $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_\ell$ are linearly ordered by gr. Finally a certain compatibility between the function gr and the multiplication of sections in $H^0(\mathcal{F}, \mathcal{L})$ is required.

Once all such hypothesis are fulfilled we are able to prove that the relations among a basis of $V^*_{\varepsilon_1} \oplus V^*_{\varepsilon_2} \oplus \cdots \oplus V^*_{\varepsilon_\ell}$ may be described in terms of the Plücker relations of $\mathcal{F}$, and for this reason we call the above general framework *plückerization* for X. Further a standard monomial theory for X may be described in terms of the standard monomial theory of $\mathcal{R}$. Using this we see that X degenerates to $\mathcal{R}$ in a G–equivariant flat way.

The construction of K, $\mathcal{F}$, $\mathcal{R}$, gr,... follows an empirical recipe. The main ingredients for this construction are suggested by the moltiplication rule of the modules $V^*_{\varepsilon_1}, V^*_{\varepsilon_2}, \ldots, V^*_{\varepsilon_\ell}$. Once these objects are defined the verifications of the above technical hypothesis are very uniform for the different varieties in the applications. In particular this recipe hints how many nodes to add to the Dynkin diagram of G in order to obtain K; for the symmetric varieties just one node while for the model varieties and another class of spherical variety (see below) two nodes are needed.

Our previous paper [4] follows the above general framework applying it to certain classes of symmetric varieties. Notice however that in that paper the proof of the existence of a standard monomial theory derived by that of the bigger group K is wrong; there we tacitly assumed that a certain map is G–equivariant and this is not the case in general. However this work amends that gap.

Our second aim is the application of the above described framework to the model varieties of type A, B and C. A homogeneous model variety for a semisimple group G is a quasi affine variety whose coordinate ring is the sum of all irreducible representations of G with multiplicity one. These varieties were introduced by Gelfand in [1] and studied by Gelfand and Zelevinsky in [6] and [7]. In particular for a homogeneous model variety G/H we have $\Omega^+ = \Lambda^+$.

In the cited papers the authors provided an embedding of the model varieties for classical groups as an open subset of a grassmannian of a bigger finite dimensional group; hence there are some similarities with our program. From the geometrical viewpoint the construction of Gelfand and Zelevinsky is more natural than our approach. However their embedding is not suitable for the application to the standard monomial theory having as generators a basis of $V^*_{\varepsilon_1} \oplus V^*_{\varepsilon_2} \oplus \cdots \oplus V^*_{\varepsilon_\ell}$. Indeed it is for this purpose that we need to use a more complicated infinite dimensional grassmannian for model varieties of type B and C. The two approaches coincide for the model variety of type A for which we use a finite dimensional lagrangian grassmannian.

Finally we study a further application of our framework to another class of spherical varieties listed as (15) in the paper [2] page 656. For this example the recipe for the construction of K is a bit different of the above reported one; this class of varieties show how our program may be applied in other cases.

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