Contact metric geometry in dimension five

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Contact metric manifolds can be considered as the odd-dimensional analogue of Hermitian manifolds, and are one of the most active research fields in differential geometry. An *almost contact structure* on a $(2n+1)$-dimensional manifold M is triple (φ, η, ξ), where ξ is a nowhere vanishing vector field, η a 1-form and φ a $(1,1)-$tensor, such that

$$\eta(\xi) = 1, \quad \varphi^2 = -\mathrm{I} + \xi \otimes \eta. \tag{1}$$

As it is well known, conditions (1) imply $\varphi(\xi) = 0$, and $\eta \circ \varphi = 0$. The vector field ξ defines the characteristic foliation $\mathcal{F}$ with one-dimensional leaves, and the kernel of η defines the codimension one sub-bundle $\mathcal{D} = \ker \eta$. Then, the tangent bundle TM of M admits the canonical splitting $TM = \mathcal{D} \oplus R\xi$. If the 1-form η satisfies the condition $\eta \wedge (\mathrm{d}\eta)^n \neq 0$, then the subbundle $\mathcal{D}$ defines a *contact structure* on M. In this case, η is called a *contact form* and the vector field ξ is called the *Reeb vector field*. If η is a contact form, then $\mathrm{d}\eta(\xi, X) = 0$, for every vector field X on M.

A Riemannian metric g on an almost contact manifold (M, φ, η, ξ) is *compatible* with the almost contact structure if

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for every vector fields X, Y. The structure (φ, η, ξ, g) is called an *almost contact metric* structure. Any almost contact structure on a paracompact manifold admits a compatible metric.

The *fundamental form* Φ associated to an almost contact metric structure (φ, η, ξ, g) is given by

$$\Phi(X, Y) = g(X, \varphi Y).$$

An almost contact metric structure (φ, η, ξ, g) is said to be *contact metric* if $2\Phi = \mathrm{d}\eta$. In this case, η is a contact form. We shall denote by (M, η, g) (or $(M, \varphi, \eta, \xi, g)$) a *contact metric manifold*, that is, an odd-dimensional manifold equipped with a contact metric structure.

Considering the product manifold $M \times R$, denoted by $(X, f\frac{d}{dt})$ an arbitrary vector field on $M \times R$, one can introduce the almost complex structure

$$J\Big(X, f\frac{\partial}{\partial t}\Big) = \Big(\varphi X - f\xi, \eta(X)\frac{\partial}{\partial t}\Big).$$

Then, (φ, η, ξ) is said to be *normal* if J is integrable. This is equivalent to requiring that the Nijenhuis tensor N_φ associated to the tensor φ satisfies the condition $N_\varphi = -\mathrm{d}\eta \otimes \xi$. A *Sasakian manifold* is a normal contact metric manifold.

A contact metric manifold $(M, \varphi, \eta, \xi, g)$ is said to be K*-contact* if the tensor $h = \frac{1}{2}\mathcal{L}_\xi \varphi$ vanishes (equivalently, if ξ is a Killing vector field). Any Sasakian manifold is K-contact, but the converse only holds in dimension three.

A contact manifold (M, η) is said to be *homogeneous* if there exists a connected Lie group G of diffeomorphisms acting transitively on M and leaving η invariant. If g is a Riemannian metric associated to η and G is a group of isometries, then (M, η, g) is said to be a *homogeneous contact metric manifold*. In this case, the whole contact metric structure (η, φ, ξ, g) is invariant.

Three-dimensional homogeneous contact metric manifolds are well understood. In fact, if (M, η, g) is a simply connected three-dimensional homogeneous contact metric manifold, then $M = G$ is a Lie group and the contact metric structure (η, g, ξ, φ) is left-invariant.

The five-dimensional case appears much broader and it allows several different interesting behaviours. The simply connected covering of a five-dimensional contact metric (locally) symmetric space is either $S^5(1)$ or $E^3 \times S^2(4)$. The classification of five-dimensional φ-symmetric spaces is known, as well as their relationship with naturally reductive spaces. Rigidity results on compact five-dimensional homogeneous contact metric manifolds have been given. It is then a natural problem to study five-dimensional homogeneous contact metric manifolds.

Five-dimensional K-contact Lie algebras. In [1], we introduced a general approach to the study of left-invariant K-contact structures on Lie groups and we obtained a full classification in dimension 5. We showed that Sasakian structures on 5-dimensional Lie algebras with non-trivial center are a relatively rare phenomenon with respect to K-contact structures.

The starting point is the following general result on the center of a contact Lie algebra: a contact Lie algebra either has trivial center, or its center is one-dimensional and spanned by the characteristic vector field.

We proved that $(2n+1)$-dimensional K-contact Lie algebras with non-trivial center are in a one-to-one correspondance with $2n$-dimensional almost Kähler Lie algebras. In fact, the contact distribution of a K-contact Lie algebra with non-trivial center is again a Lie algebra, which inherits an almost Kähler structure. Conversely, K-contact Lie algebras with non-trivial center are constructed in a natural way as contactizations of almost Kähler Lie algebras. With regard to $(2n+1)$-dimensional K-contact Lie algebras with trivial center, we showd that if $n \geq 2$ then $\ker ad_\xi$ is a K-contact Lie subalgebra with non-trivial center.

These results permit us to understand the structure of K-contact (not Sasakian) Lie algebras, both with trivial and non-trivial center, in any dimension $2n+1 \geq 5$, and to obtain the full classification of five-dimensional ones.

Up to isomorphisms, there exist 11 types of 5-dimensional K-contact non-Sasakian Lie algebras with non-trivial center. Moreover, each Sasakian Lie algebra with non-trivial center also admits infinitely many K-contact structures which are not Sasakian. We also showed that a 5-dimensional solvmanifold with a left-invariant K-contact (not Sasakian) structure is a S^1-bundle over a symplectic solvmanifold.

If a 5-dimensional Lie algebra with trivial center admits a K-contact structure, then the K-contact structure is necessarily Sasakian.

We also discussed some curvature properties of 5-dimensional K-contact and Sasakian structures, proving that a 5-dimensional K-contact η-Einstein Lie algebra is necessarily Sasakian and classifying 5-dimensional φ-symmetric Sasakian Lie algebras. We also consider hypo structures, proving that a 5-dimensional K-contact hypo Lie algebra is Sasakian and η-Einstein.

Finally, we considered 5-dimensional K-contact and Sasakian pseudo-metric structures, allowing the associated metric to be of any signature. While the classification of five-dimensional Lie algebras with non-trivial center admitting a K-contact pseudo-metric structure coincides with the one of K-contact Lie algebras, there exist four types of five-dimensional Lie algebras with non-trivial center, which do not admit any Sasakian structure but do admit Sasakian pseudo-metric structures.

Contact metric structures on five-dimensional generalized symmetric spaces.

A *φ-symmetric space* may be considered as the odd-dimensional analogue of a Hermitian symmetric space. In fact, it is a Sasakian manifold $(M, \varphi, \eta, \xi, g)$, such that the geodesic reflections with respect to the integral curves of ξ (φ-geodesic symmetries) extend to define global automorphisms of the entire structure. The existence of φ-geodesic symmetries yields that the manifold fibers over a Hermitian symmetric space.

A well known result states that a simply connected and complete locally φ-symmetric space is naturally reductive. Conversely, five-dimensional naturally reductive spaces carrying a structure of φ-symmetric space were completely classified by O. Kowalski and S. Wegrzynowski. These examples are generalized symmetric spaces.

Given a connected pseudo-Riemannian manifold (M, g) and x a point of M, a symmetry at x is an isometry s_x of M, having x as isolated fixed point. When (M, g) is a symmetric space, each point x admits a symmetry s_x reversing geodesics through the point. Hence, s_x is involutive for all x. Generalizing this property, A.J. Ledger defined a *regular s-structure* as a family $\{s_x : x \in M\}$ of symmetries of (M, g), satisfying, for all points x, y of M,

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

The *order* of an s-structure is the least integer $k \geq 2$, such that $(s_x)^k = id_M$ for all x (it may be that $k = \infty$). A *generalized symmetric space* is a connected pseudo-Riemannian manifold (M, g) admitting a regular s-structure. The order of a generalized symmetric space is the infimum of all integers $k \geq 2$ such that M admits a regular s-structure of order k.

Five-dimensional Riemannian generalized symmetric spaces are classified into 12 classes of homogeneous manifolds. Comparing this classification list with the classification of five-dimensional naturally reductive spaces, it is easily seen that the generalized symmetric spaces which are not naturally reductive are the ones of type $2, 3, 4, 7, 8a, 8b$ (all of order 4) and 9 (of order 6).

The results of O. Kowalski and S. Wegrzynowski on φ-symmetric and naturally reductive spaces lead to the following

QUESTION 1: *Do there exist invariant contact metric structures on five-dimensional generalized symmetric spaces which are not naturally reductive?*

QUESTION 2: *Besides the structures of globally φ-symmetric spaces, do there exist other invariant contact metric structures on five-dimensional generalized symmetric spaces which are naturally reductive?*

Questions 1 and 2 have been completely answered in [2], classifying all invariant contact metric structures on five-dimensional generalized symmetric spaces. With regard to the examples which are not naturally reductive, while several of them do not carry any invariant contact metric structure, we find and explicitly describe four new families of homogeneous contact metric structures, on five-dimensional generalized symmetric spaces of type $3, 8a, 8b$ and 9.

These homogeneous contact metric manifolds are not Sasakian (not even K-contact), but belong to the wider class of *H-contact manifolds*, that is, their Reeb vector field ξ is a critical point for the energy functional restricted to the space $\chi^1(M)$ of all unit vector fields. Einstein and η-Einstein invariant contact metric structures have also been pointed out.

On the other hand, a rigidity result was obtained for the naturally reductive cases, as it turned out that the only invariant contact metric structures on the naturally reductive examples, are the ones corresponding to globally φ-symmetric spaces.

REFERENCES

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