Geometry of homogeneous pseudo-Riemannian manifolds

G. Calvaruso, [1] A. Fino, [2] M. Brozos-Vazquez,[3] Garcia-Rio,[3] S. Gavino-Fernandez,[3]

[1]Dipartimento di Matematica e Fisica "E. De Giorgi", Università del Salento, Italy

[2]Dipartimento di Matematica, Università di Torino, Italy

[3]Universidad de Santiago de Compostela, Spain

A pseudo-Riemannian manifold (M, g) is *(locally) homogeneous* if for any two points $p, q \in M$, there exists a (local) isometry ϕ, mapping p to q. Homogeneous and locally homogeneous manifolds are among the most investigated object in Differential Geometry, also for their physical applications. In this framework, natural problems are to classify all homogeneous pseudo-Riemannian manifolds (M, g) of a given dimension, and to determine examples with special geometric properties. This problem has been intensively studied, especially in the low-dimensional cases.

Ricci solitons. Ricci solitons were introduced by Hamilton and they are a natural generalization of Einstein metrics. A pseudo-Riemannian metric g on a smooth manifold M is called a *Ricci soliton* if there exists a a smooth vector field X, such that

$$\mathcal{L}_X g + \varrho = \lambda g, \qquad (1)$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, ϱ denotes the Ricci tensor and λ is a real number. A Ricci soliton g is said to be a *shrinking*, *steady* or *expanding* according to whether $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively.

Ricci solitons are the self-similar solutions of the *Ricci flow* and are important in understanding its singularities. The interest in Ricci solitons has also risen among theoretical physicists in relation with String Theory. After their introduction in the Riemannian case, the study of pseudo-Riemannian Ricci solitons attracted a growing number of authors (see for instance the references inside [1] and [2]).

If $M = G/H$ is a homogeneous space, a *homogeneous Ricci soliton* on M is a G-invariant metric g for which equation (1) holds. In particular, by an *invariant Ricci soliton* we mean a homogeneous one, such that equation (1) is satisfied by an invariant vector field.

It is a natural question to determine which homogeneous manifolds G/H admit a G-invariant Ricci soliton. All known examples of homogeneous Riemannian Ricci soliton metrics on non-compact homogeneous manifolds are isometric to some solvsolitons, that is, to invariant Ricci solitons on a solvable Lie group.

The difference between Riemannian and pseudo-Riemannian settings lead to different results concerning the existence of homogeneous Ricci solitons.

With regard to the three-dimensional case, although there exist three-dimensional Riemannian homogeneous Ricci solitons, a strong rigidity result holds: there are no left-invariant Ricci solitons on three-dimensional Riemannian Lie groups. On the other hand, left-invariant Ricci solitons on three-dimensional Lorentzian Lie groups were classified in [1]. Indeed, the three-dimensional Lorentzian case allows the existence of expanding, steady and shrinking left-invariant Ricci solitons.

By a previous result of the first author, three-dimensional locally homogeneous Lorentzian manifolds are either locally symmetric or locally isometric to a three-dimensional Lie group equipped with a left-invariant Lorentzian metric. Locally symmetric Lorentzian three-manifolds which are not of constant sectional curvature are either locally isometric to a Lorentzian product of a real line and a surface of constant Gauss curvature, or they are Walker manifolds with a two-step nilpotent Ricci operator. It is clear that, in addition to Einstein spaces, products $N^k(c) \times R$ with $N^k(c)$ of constant sectional curvature are Ricci solitons, in both the Riemannian and the Lorentzian case. By a *non-trivial* Ricci soliton, one means a Ricci soliton which is neither Einstein nor a product $N^k(c) \times R$. Ricci solitons on Walker manifolds were considered in [1], proving the existence of expanding, steady and shrinking locally symmetric Ricci solitons.

From the explicit classifications obtained in [1], the following geometric characterization follows, in terms of the canonical form (*Segre type*) of the Ricci operator:

THEOREM: A complete and simply connected three-dimensional homogeneous Lorentzian manifold is a non-trivial Ricci soliton if and only if the Ricci operator $\hat{Ric}$ is not diagonalizable and has exactly three equal eigenvalues, that is, $\hat{Ric}$ is either of Segre type $\{3\}$ or of degenerate Segre type $\{21\}$.

A homogeneous pseudo-Riemannian manifold (M, g) is said to be *reductive* if $M = G/H$ and the Lie algebra $\mathbf{g}$ can be decomposed into a direct sum $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$, where $\mathbf{m}$ is an $\mathrm{Ad}(H)$-invariant subspace of $\mathbf{g}$. It is well known that when H is connected, this condition is equivalent to the algebraic condition $[\mathbf{h}, \mathbf{m}] \subseteq \mathbf{m}$. In the study of homogeneous pseudo-Riemannian manifolds, a fundamental difference arises between the Riemannian case and the non Riemannian one. In fact, while any homogeneous Riemannian manifold is reductive, in dimension four and higher there exist homogeneous pseudo-Riemannian manifolds which do not admit any reductive decomposition.

Four-dimensional Ricci solitons on non-reductive homogeneous pseudo-Riemannian manifolds were classified in [2], for solutions of (1) determined by vector fields $V \in \mathbf{m}$. Non-trivial examples appear both in the Lorentzian case and for metrics of neutral signature $(2, 2)$.

Other aspects of the geometry of four-dimensional non-reductive homogeneous pseudo-Riemannian four-manifolds were studied in [2] as well. In particular, invariant complex structures were classified, and it was proved that four-dimensional non-reductive homogeneous pseudo-Riemannian manifolds do not admit non-trivial pseudo-Kähler homogeneous Ricci solitons.

Harmonicity of vector fields on four-dimensional generalized symmetric spaces.

Let (M, g) be a connected pseudo-Riemannian manifold and x a point of M. A symmetry at x is an isometry s_x of M, having x as isolated fixed point. When (M, g) is a symmetric space, each point x admits a symmetry s_x reversing geodesics through the point. Hence, s_x is involutive for all x. Generalizing this property, A.J. Ledger defined a *regular s-structure* as a family $\{s_x : x \in M\}$ of symmetries of (M, g), satisfying, for all points x, y of M,

$$s_x \circ s_y = s_z \circ s_x, \quad z = s_x(y).$$

The *order* of an s-structure is the least integer $k \geq 2$, such that $(s_x)^k = id_M$ for all x (it may be that $k = \infty$). A *generalized symmetric space* is a connected pseudo-Riemannian manifold (M, g) admitting a regular s-structure. The order of a generalized symmetric space is the infimum of all integers $k \geq 2$ such that M admits a regular s-structure of order k.

Generalized symmetric spaces have been intensively studied under different points of view (see for example the References in [3]). In [3], harmonicity properties of vector fields on four-dimensional pseudo-Riemannian generalized symmetric spaces have been studied.

Consider a (smooth, oriented, connected) n-dimensional pseudo-Riemannian manifold (M, g), and its tangent bundle TM, equipped with the *Sasaki metric* g^s (also referred to as the *Kaluza-Klein metric* in Mathematical Physics). Given a smooth vector field V on M, the *energy* of V is, by definition, the energy of the corresponding smooth map $V : (M, g) \to (TM, g^s)$, that is,

$$E(V) = \frac{1}{2} \int_M (\mathrm{tr}_g V^* g^s) dv$$

(for M compact; in the non-compact case, one works over relatively compact domains). If $V : (M, g) \to (TM, g^s)$ is a critical point for the energy functional, then V is said to *define a harmonic map*. A vector field V defines a harmonic map if and only if its *tension field* $\tau(V) = \mathrm{tr}(\nabla^2 V)$ vanishes, that is,

$$\mathrm{tr}[R(\nabla.V, V)\cdot] = 0 \quad \text{and} \quad \nabla^*\nabla V = 0,$$

where with respect to a pseudo-orthonormal local frame $\{e_1, .., e_n\}$ on (M, g), with $\varepsilon_i = g(e_i, e_i) = \pm 1$ for all indices i, one has

$$\nabla^*\nabla V = \sum_i \varepsilon_i \left(\nabla_{e_i} \nabla_{e_i} V - \nabla_{\nabla_{e_i} e_i} V \right).$$

When g is Riemannian and M is compact, the only vector fields defining harmonic maps are the parallel ones. The weaker condition that $V : (M, g) \to (TM, g^s)$ is a critical point for the energy functional $E|_{\mathcal{X}(M)}$, restricted to maps defined by vector fields, leads again to parallel vector fields.

The existence of parallel vector fields is a rare phenomenon, which has strong consequences on the manifold itself. In particular, a Riemannian manifold admitting a parallel vector field is locally reducible, and the same is true for a pseudo-Riemannian manifold admitting an either space-like or time-like parallel vector field. This led to consider different situations, where some interesting types of non-parallel vector fields can be characterized in terms of harmonicity properties.

Let $\rho \neq 0$ be a real constant and $\mathcal{X}^\rho(M) = \{W \in \mathcal{X}(M) : ||W||^2 = \rho^2\}$. Then, one can consider vector fields $V \in \mathcal{X}^\rho(M)$ which are critical points for the energy functional $E|_{\mathcal{X}^\rho(M)}$, restricted to vector fields of the same constant length. The Euler-Lagrange equations of this variational condition is

$$\nabla^*\nabla V \quad \text{is collinear to} \quad V.$$

This characterization, first proved in the Riemannian case, remains valid in pseudo-Riemannian settings, provided that $\rho \neq 0$, that is, if V is not light-like. Although the case of light-like vector fields is more delicate to deal with, if V is a light-like vector field then the above equation is still a sufficient condition so that V is a critical point for the energy functional $E|_{\mathcal{X}^0(M)}$, restricted to light-like vector fields.

In pseudo-Riemannian settings, the same critical point conditions turn out to be much less restrictive than for Riemannian manifolds and can be satisfied by some interesting non-parallel vector fields. In [3], harmonicity properties of vector fields on four-dimensional pseudo-Riemannian generalized symmetric spaces have been studied. These spaces, classified into four different types, are good candidates for such an investigation, as their Levi-Civita connection and curvature, although rather simple to describe, are far from trivial. In particular, in most of the cases, no parallel vector fields occur. Let $M = G/H$ be a four-dimensional pseudo-Riemannian generalized symmetric space, and $\mathbf{g} = \mathbf{h} \oplus \mathbf{m}$ the corresponding decomposition of the Lie algebra $\mathbf{g}$ of G. Once applied to vector fields belonging to $\mathbf{m}$, the above conditions translate into some systems of algebraic equations for the components of these vector fields. Some interesting behaviours are found. Examples corresponding to invariant vector fields have been pointed out. Vector fields which also define harmonic maps have been completely classified. Finally, the energy of all these vector fields has been explicitly calculated.

REFERENCES

1. M. Brozos-Vazquez, G. Calvaruso, E. Garcia-Rio and S. Gavino-Fernandez, *Three-dimensional Lorentzian homogeneous Ricci solitons*, Israel J. Math., 188 (2012), 385–403.
2. G. Calvaruso and A. Fino, *Ricci solitons and geometry of four-dimensional non-reductive homogeneous spaces*, Canadian J. Math., 64 (2012), 778–804.
3. G. Calvaruso, *Harmonicity of vector fields on four-dimensional generalized symmetric spaces*, Central Eur. J. Math., 10 (2012), 411–425.