# The Price of Stability for Undirected Broadcast Network Design with Fair Cost Allocation is Constant

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*Congestion games* [8] are a well established approach to model resource sharing among selfish players. In such games, a set of resources is available to a set of $n$ players. Every player comes along with a set of strategies, each corresponding to the selection of a subset of resources. A state of the game is any combination of strategies for the players. The cost incurred by a player in a given state is defined as the sum of the costs associated to each selected resource, which depends on the number of players choosing it. The *social cost* of a state denotes its quality from a global perspective, which is typically defined as the sum of the players' costs or the maximum among the players' costs. By defining an elegant potential function, Rosenthal [8] has shown that the natural decentralized mechanism known as Nash dynamics, in which at each step some player performs an improving deviation by switching her strategy to a better alternative, is guaranteed to converge to a (pure) *Nash equilibrium* [7], i.e., a fixed point of such dynamic in which no player can improve her situation by unilaterally changing her selected strategy. A Nash equilibrium may not necessarily minimize the social cost. A widely used measure for quantifying the quality of equilibria, and thus the performance degradation due to the players' selfish behavior, is the *price of anarchy*, introduced by Koutsoupias and Papadimitriou [5], which is formally defined as the worst-case ratio of the social cost of a Nash equilibrium to the optimal social cost.

*Network design games with fair cost allocations*, introduced by Anshelevich *et al.* [1], are one of the most interesting subclasses of congestion games. In such games, we are given a graph with non-negative edge costs and, for each player, a source and a destination node. The goal of a player is to choose a path connecting her source and destination nodes. Thus, edges correspond to resources and paths connecting source and destination nodes to strategies (subsets of resources). The cost of each edge $e$ is equally shared by all the players whose selected path contains $e$, i.e., according to the Shapley value [9]. A relevant and largely investigated special case of network design games occurs when all players share the same source node (*multicast games*). In this case, players are assumed as being associated to the endpoint node they wish to connect with the source. *Broadcast games* are multicast games in which there is a player associated to every node of the network.

In their seminal paper, Anshelevich *et al.* [1] raised the problem of the bad performance of Nash equilibria in network design games. The price of anarchy, in fact, is as large as the number of players even for broadcast games in undirected graph. Motivated by this issue, they started to explore the middle ground between centrally enforced solutions and completely unregulated anarchy by proposing the notion of *price of stability* (PoS), that is the ratio of the social cost of the cheapest Nash equilibrium and the social cost of an optimal solution. They argued that each local minimum of Rosenthal's potential function is a Nash equilibrium and, by comparing the social cost of the global minimum with that of an optimal solution, they obtained an upper bound of $H_n := \sum_{i=1}^{n} 1/i = O(\log n)$ on the PoS of network design games. They also provided an instance of broadcast games in directed graphs for which PoS $= H_n$, thus completely characterizing the PoS of network design games in the directed case. However, since then, the question of determining tight bounds for the case of undirected graphs has stood as a major open problem and after all these years is still far from being solved.

At the time moment, while no improvements on the $O(\log n)$ result by Anshelevich *et al.* [1] have been achieved for network design games, two upper bounds of $O(\log \log n)$ and $O(\log n/\log \log n)$ have been given by Fiat *et al.* [3] for broadcast games and by Li [6] for multicast games, respectively. However, the best-known lower bounds, determined by Bilò *et al.* [2], are 1.818 for broadcast games, 1.862 for multicast games and 2.245 for network design games, thus leaving a huge gap to be filled.

A recent result by Kawase and Makino [4] shows that, even in broadcast games, the social cost of the Nash equilibrium minimizing Rosenthal's potential function, which is at the basis of Anshelevich *et al.*'s approach, can be $\Omega(\sqrt{\log\log n})$ times the cost of the social optimum. This implies that, in order to get an $o(\sqrt{\log\log n})$ upper bound on the PoS, one has to resort on different arguments.

In this work, we close the PoS question for broadcast games by proving the following result.

**Theorem 0.1.** *The* PoS *of broadcast games in undirected graphs is* $O(1)$.

Such a result is achieved by introducing and exploiting the new concept of *homogenization*. Roughly speaking, a state is homogeneous with respect to an optimal state $T^*$ if the difference between the costs of any two players is upper bounded by a certain function of the set of edges connecting them in $T^*$. We call *homogenization process* a transformation that has the property of decreasing Rosenthal's potential starting from a given non-homogeneous state. The nice property possessed by homogeneous states is that, for each improving deviation by a player that causes the insertion of an edge $e$ not belonging to $T^*$, there always exists either a subsequent improving deviation which immediately removes $e$ from the state, or a sequence of improving deviations, that we call *absorbing process*, which is able to attract a consistent part of $T^*$ in the current state. Thanks to the afore mentioned properties, it is possible to design an algorithm which, starting from $T^*$, suitably combines improving deviations, homogenization and absorbing processes so as to generate a sequence of states which ends up at a Nash equilibrium whose social cost compares nicely with that of $T^*$.

We stress here that the idea of constructing a Nash equilibrium of small social cost as an output of an algorithm that suitably schedules a sequence of improving deviations starting from an optimal state was already at the basis of Fiat *et al.*'s approach [3]. Our approach, however, is not a refinement of their technique, as it strongly relies on the new properties of homogeneous profiles. Moreover, our homogenizing process does not consist of improving deviations, but it corresponds to a transformation globally decreasing the potential. Hence, it can be appreciated how crucial is the role of the novel concept of homogenization in the process of lowering the PoS from a super-constant to a constant factor.

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