The Complexity of Decision Problems about Nash Equilibria in Win-Lose Games

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Among the most fundamental problems in *Algorithmic Game Theory* are those concerning the *Nash equilibria* of a *strategic game*: states where no *player* could unilaterally deviate to improve her *utility.* Such algorithmic problems, including their *decision*, *search* and *approximation* variants, have been studied extensively in the last few years. The fundamental theorem of Nash [4,5] that Nash equilibria are guaranteed to exist makes the search problem for Nash equilibria *total*, which implies that the search problem is *not* $\mathcal{NP}$-complete unless $\mathcal{NP} = co\text{-}\mathcal{NP}$.

Decision problems about Nash equilibria result naturally by twisting the search problem in one of several simple ways that deprive it from its existence guarantees. Here is a (non-exhaustive) list of decision problems about Nash equilibria: Given a strategic game, does it have: *(i)* A Nash equilibrium where each player has utility at least a given number? [3],*(ii)* A Nash equilibrium where each player has utility at most a given number?, *(iii)* At least two Nash equilibria? [3], *(iv)* A Nash equilibrium whose support contains a set of strategies? [3], *(v)* A Nash equilibrium whose support is contained in a set of strategies? [3], *(vi)* A Nash equilibrium whose support has size greater than a given number? [3], *(vii)* A Nash equilibrium whose support has size smaller than a given number? [3], *(viii)* A Nash equilibrium in which the total utility of players is at least a given number? [2], *(ix)* A Nash equilibrium in which the total utility of players is at most a given number?, *(x)* A *rational* Nash equilibrium (i.e., one with all probabilities rational)? [1].

Some of these decision problems are $\mathcal{NP}$-complete for *symmetric* two-player games; this was originally shown by Gilboa and Zemel [3] and later by Conitzer and Sandholm [2] via a unifying reduction from the *satisfiability* problem (which covered some additional decision problems over those considered in [3]). The last problem in the list is $\mathcal{NP}$-complete even for three-player games [1] — recall that all Nash equilibria of a two-player game are rational, so that the problem is trivial for two-player games.

In this work, we settle the complexity of the natural decision problems about Nash equilibria previously considered in [2,3] (or introduced here) for win-lose games, i.e., games in which all utilities are 0 *or* 1. Specifically, we show, as our main result, that these decision problems are $\mathcal{NP}$-complete for two-player win-lose games. In a similar vein, the decision problem asking whether a given game has a rational Nash equilibrium [1] is shown $\mathcal{NP}$-complete for three-player win-lose games. Thus, these decision problems about Nash equilibria have the same complexity for win-lose games as for general games.

To show our results, we first prove a significant milestone, which we describe. Say that a game has the ***positive utility property*** if each player has always a response to the choices of the other players that makes her utility greater than zero. Note that for two-player win-lose games, the positive utility property implies that the utility matrix of the row player (resp., column player) cannot have a column (resp., row) containing only zeros. We revisit the decision problem from [1] asking whether a given game has the same set of Nash equilibria with a ***gadget game***, and additionally we assume that the gadget game has the positive utility property; we show that, *when restricted to win-lose games*, this problem is co-$\mathcal{NP}$-hard for any choice of a win-lose gadget game (with the positive utility property).

In the backbone technical result of our research, we utilizes a reduction from the satisfiability problem, which establishes that the *unsatisfiability* of a given formula is equivalent with the fact that the constructed win-lose game does not have a Nash equilibrium with properties opposite to those possessed by the Nash equilibria of the gadget game, while the *satisfiability* of the formula always guarantees the existence of at least one Nash equilibrium with some particular properties which are independent from both the formula and the gadget game (i.e., they hold *a priori* as a feature of the reduction). This implies, in particular, that deciding whether the two win-lose games have the same set of Nash equilibria is co-$\mathcal{NP}$-hard, improving the result in [1] which applies to general two-player games.

The reduction used for the proof of our major theorem constitutes a major improvement over previous reductions from the satisfiability problem in [1,2] to **yield a win-lose game** (rather than a general game) **while preserving** the relation between its Nash equilibria and the satisfiability of the formula.

Moreover, by suitable choices of the gadget game so that the properties possessed by its Nash equilibria dismatch the properties of the Nash equilibria induced when the formula is satisfiable, particular $\mathcal{NP}$-hardness results follow. These results extend the corresponding results from [2,3] which apply to two-player symmetric games with rational utilities.

For example, choosing the gadget game as a two-player win-lose game where each player has a single strategy and all utilities are 1 implies that a handful of properties are $\mathcal{NP}$-hard to decide for two-player, win-lose games. Choosing the gadget game as a three-player win-lose game with a single irrational Nash equilibrium implies that deciding the existence of a rational Nash equilibrium is $\mathcal{NP}$-hard for three-player win-lose games.

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