New Bounds for the Balloon Popping Problem

Davide Bilò[1] and Vittorio Bilò[2]

[1]Dipartimento di Teorie e Ricerche dei Sistemi Culturali, Università di Sassari, Italy

[2]Dipartimento di Matematica e Fisica "Ennio De Giorgi", Università del Salento, Italy

In digital goods auctions, also known as unlimited supply auctions, an auctioneer sells a collection of identical items to unit-demand bidders. As usual in this setting, when getting the item, a bidder experiences a certain utility which is unknown to the auctioneer. Given this restriction, the auctioneer wants to design a mechanism, i.e., a set of auction rules, so as to raise as much revenue as possible from the bidders. The competitiveness of a certain mechanism is usually measured with respect to the maximum revenue that can be raised by an omniscient auctioneer who is restricted to offer the goods at the same price to all bidders (*fixed-price scheme*). One can ask whether this is a reasonable restriction, and whether without this restriction, the auctioneer can achieve a considerably higher revenue.

Immorlica *et al.* [1] provided an answer to this question for the case of *ascending auctions* with *anonymous bidders*. More precisely, they consider the scenario in which *i)* the auctioneer knows the set of the bidders' utilities, but is unable to determine which bidder has which utility, and *ii)* during the auction, the auctioneer can only rise the price offered to a bidder. In particular, Immorlica *et al.* [1] show that, under conditions *i)* and *ii)*, no mechanism can raise a revenue of more than a constant time the one raised by the best fixed-price scheme.

Such a result is achieved by introducing and studying the *balloon popping problem* which is defined as follows.

We are given n undistinguishable balloons of capacities $1, 1/2, 1/3, \ldots, 1/n$ and we are asked to blow them so as to maximize the total volume of air inflated in the balloons knowing that a balloon blown up beyond its capacity pops (and reveals its capacity) thus giving no contribution to the total volume. What is the best blowing strategy and what total volume is achievable?

A blowing strategy is said to be *offline* if it has the chance to get back to an already-inflated balloon and inflate it further, while it is said to be *online* if it has to process the balloons sequentially, but is granted the knowledge of the balloon's capacity as soon as it is processed, regardless of whether or not it popped. Hence, an online blowing strategy must take irrevocable decisions in a scenario in which it has full knowledge of the set of capacities of the remaining balloons, while an offline blowing strategy has no particular restrictions on how to process the balloons, but it has to operate in a scenario with incomplete knowledge. Let $\mathbf{OFF}_n$ and $\mathbf{ON}_n$ be the *expected* total volume achievable by the optimal offline and online blowing strategy, respectively. For any fixed integer $n \geq 1$, denote as $\mathcal{Y}_k$ the family of all subsets of $\{1, 1/2, \ldots, 1/n\}$ of size k, where each $Y \in \mathcal{Y}_k$ is of the form $\{1/y_1, \ldots, 1/y_k\}$, with $y_1 < \ldots < y_k$; so $Y \in \mathcal{Y}_k$ is a set of k balloons listed in decreasing order.

Immorlica *et al.* [1] prove that under conditions *i)* and *ii)* stated above, for any set of n bidders with arbitrary utilities, the *expected revenue* of any mechanism is at most $\mathbf{OFF}_n$ times the one raised by the best fixed-price scheme. In order to upper bound $\mathbf{OFF}_n$, they first show that $\mathbf{OFF}_n \leq \mathbf{ON}_n$ for any positive integer n, then they determine the optimal online blowing strategy, thus proving that

$$\mathbf{ON}_n = \sum_{k=1}^{n} \sum_{Y \in \mathcal{Y}_k} \frac{\max_{1 \leq j \leq k}\{j/y_j\}}{\binom{n}{k} \cdot k}, \tag{1}$$

and finally they show that $\mathbf{ON}_\infty \leq 4.331$. Moreover, they prove that a *greedy mechanism*, that is, the offline blowing strategy which tries to blow up each balloon at the maximum possible capacity, achieves an expected volume equal to $\sum_{i=1}^{n} 1/i^2$ which implies $\mathbf{OFF}_\infty \geq \pi^2/6 \approx 1.6449$.

Jung and Chwa [2] improved these bounds by designing an offline blowing strategy, called *Bunch*, yielding $\mathbf{OFF}_\infty \geq 1.6595$ and proving that the right hand side of Equation (1) is at most $2 - H_n/n$, which yields $\mathbf{ON}_\infty \leq 2$. They also formulate a conjecture which, whenever true, would give $\mathbf{ON}_n \geq 2-(2H_n-1)/n$, thus implying $\mathbf{ON}_\infty = 2$.

We further improve the lower bound on $\mathbf{OFF}_\infty$ by designing an offline blowing strategy, that we call *Group*$_k$. This strategy works by partitioning the $n-1$ smallest balloons into $(n-1)/k$ groups of size k and then applying an ad-hoc (possibly optimal) strategy to the first of these groups and a simple basic blowing strategy to each of the remaining ones. The performance of $Group_k$ improves when k increases and better lower bounds can be achieved by delving into a more detailed analysis covering higher values of k. For $k = 5$, we achieve $\mathbf{OFF}_\infty \geq 1.68$.

We also focus on the exact computation of $\mathbf{ON}_n$ for any integer $n \geq 1$. To this aim, note that the characterization of $\mathbf{ON}_n$ given by Immorlica *et al.* [1] through Equation (1) is neither easy to be analytically analyzed nor efficiently computable, since it requires to generate all subsets of a set of cardinality n whose number is exponential in n. We give an alternative exact formula for $\mathbf{ON}_n$ which, although remaining of challenging analytical analysis, can be computed by an algorithm of running time $O(n^5)$. This allows us to disprove the conjecture formulated by Jung and Chwa [2] (which was experimentally verified only for $n \leq 21$) as soon as $n \geq 44$ and to provide an empirical evidence that $\mathbf{ON}_\infty < 2$. We hence **conjecture** that

$$\mathbf{ON}_n \leq \sum_{i=1}^{n} \frac{1}{i^2} + \frac{1}{2n} \sum_{i=2}^{n} \left(\frac{n-i}{n} - \frac{1}{i+1} \right). \qquad (2)$$

The validity of this new conjecture, which thanks to our algorithm can be verified for n in the order of the hundreds, would imply $\mathbf{ON}_\infty \leq \pi^2/6 + 1/4 \approx 1.8949$. Such a value matches an experiment conducted by Immorlica *et al.* [1] who estimated $\mathbf{ON}_{1000}$ by applying the optimal online blowing strategy on a sample of 10^5 random sequences of balloons.

REFERENCES

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