Wave front sets for ultradistribution solutions of linear partial differential operators with coefficients in non–quasianalytic classes

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The ultradifferentiable functions are intermediate classes between real analytic and C^∞ functions. Depending on their topological structure they can be classified as classes of Beurling or of Roumieu type. Moreover, according to the Denjoy-Carleman theorem the ultradifferentiable functions can also be classified in quasianalytic and non-quasianalytic classes, where, roughly speaking, the quasianalytic ones are functions whose Fourier transform has a stronger decay at infinity. We refer to [2,8,7,3] for the different ways to introduce these classes. We emphasize that the Gevrey classes are particular cases of non-quasianalytic ultradifferentiable functions of Roumieu type.

We establish some basic results on propagation of singularities for solutions of linear partial differential operators with coefficients in certain smooth classes, in a wider setting than the one of classical distributions or, even, ultradistributions of Gevrey type, and from the micro-local point of view, i.e., via wave front sets. The notion of wave front set was introduced by Hörmader in 1970 to simplify the study of the propagation of singularities of (ultra)distribution solutions of linear partial differential operators.

It is known that partial differential operators with ultradifferentiable coefficients, or even pseudodifferential operators of ultradifferentiable type, reduce in most cases the singular support and the wave front set of ultradistributions. The main aim of this paper is to prove a suitable converse result, more precisely, we prove that if $P = P(x, D)$ is a linear partial differential operator with ultradifferentiable coefficients on $\Omega \subset \mathbb{R}^N$ then the following inclusion holds

$$WF_\omega(u) \subset WF_\omega(Pu) \cup \Sigma, \qquad (0.1)$$

for all ultradistributions with compact support $u \in \mathcal{E}'_\omega(\Omega)$, where Σ is the characteristic manifold of P, i.e., the set where the principal symbol of P vanishes. We cover both Beurling and Roumieu cases in inclusion (0.1). The analogous inclusion for the C^∞-wave front set and classical distributions was proved by Hörmander [6]. Actually, Hörmander [7] established the validity of inclusion (0.1) also for wave front sets of classical distributions in the setting of certain quasianalytic (and non-quasianalytic) classes of Roumieu type including the spaces of real analytic functions. On the other hand, the present authors extended (0.1) for wave front sets of classical distributions to the Beurling and Roumieu cases at the same time in [1] when the weight functions (quasianalytic or not) and the corresponding spaces of ultradifferentiable functions are considered from the point of view of Braun, Meise and Taylor [3]. We point out that results of this type are a powerful tool in the study of the problem of iterates from a micro-local point of view. Here we mainly deal with non-quasianalytic classes of Beurling type in the sense of Braun, Meise and Taylor [3], and we prove inclusion (0.1) for ultradistributions of Beurling (and of Roumieu) type with compact support. Since the growth of the Fourier transform of compactly supported ultradistributions of Beurling (or even Roumieu) type is rather different to the one of classical distributions, we cannot use here the techniques of Hörmander [7], as we did in [1]. Thus, we need to follow a different approach. Precisely, we use some techniques coming from the theory of pseudodifferential operators of Beurling type and infinite order developed in [4,?]; observe that in non-quasianalytic classes the use of test functions is allowed and hence, the machinery of [4,?] can be used. We obtain first the Beurling case. Then, using [5, Proposition 2], a comparison between Beurling and Roumieu wave front sets (such a comparison has been recently extended in [1] also to cover the case of quasianalytic weight functions), we obtain immediately the corresponding Roumieu version of inclusion (0.1). Because of the topology, in the Beurling case we need to take the coefficients of the operator P in a smaller class. This is not the case in the Roumieu setting. In both cases we need weight functions that are equivalent to sub-additive weight functions. An extension to classical properly supported pseudodifferential operators of the inclusion (0.1) is also obtained. Finally, we mention that the different properties that we prove on pseudodifferential operators are obtained, as in [5], avoiding the difficult techniques of wave front sets of kernels (as it is usual in the literature; see for example, [9, Section 3.4] or [6, Chapter VIII]). An application to the study of wave front sets of solutions of partial differential operators is provided.

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