Mean Ergodic semigroups of operators

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For a continuous linear operator T in a Banach space X, its Cesàro means are defined by

$$T_{[n]} := \frac{1}{n} \sum_{m=0}^{n-1} T^m, \quad n \in \mathbb{N}. \tag{0.1}$$

If $\{T_{[n]}\}_{n=1}^{\infty}$ is convergent in the strong operator topology, then T is called *mean ergodic*. The interest in such operators has its origins in statistical mechanics and probability theory. In such settings, one also considers continuous processes ϕ_t, with t specifying time, which in many situations satisfy $\phi_t(\phi_s(u)) = \phi_{t+s}(u)$ for all points u in a phase space and all times s, t. The abstract setting then consists of a *semigroup* of continuous linear operators $(T(t))_{t\geq 0}$ acting in X (i.e., $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$) and one investigates the long term behaviour of $(T(t))_{t\geq 0}$ via its Cesàro averages $C(r) := r^{-1} \int_0^r T(t)dt$ for $r > 0$. If this limit exists in the strong operator topology, then $(T(t))_{t\geq 0}$ is called *mean ergodic*. Fix $t_0 > 0$. Given $n \in \mathbb{N}$ it turns out that $C(nt_0) = T(t_0)_{[n]}C(t_0)$ and one sees the connection between the Cesàro averages $\{C(nt_0)\}_{n=1}^{\infty}$ of the semigroup $(T(t))_{t\geq 0}$ and the discrete Cesàro means $\{T(t_0)_{[n]}\}_{n=1}^{\infty}$ of the individual operator $T(t_0)$. So, there is a need to simultaneously investigate individual mean ergodic operators and mean ergodic systems of operators (i.e., semigroups). This has happened ever since the origins of the subject (i.e., the 1930's).

Much of modern analysis occurs in locally convex Hausdorff spaces (briefly, lcHs) which are *non–normable*. So, there is an interest in extending the Banach space results for mean ergodicity to this setting. For individual operators (especially Banach space results related to the theory of bases, [7]) this has been carried out in recent years, [1], [2], [3], [4], [5], [9], [10]. For certain aspects of the theory of mean ergodic *semigroups* of operators in the non–normable setting we refer to [6], [8, Ch.2], [11, Ch.III, §7], and the references therein. The aim of this paper is to develop this topic further, also in some rather different directions.

The natural framework is a sequentially complete lcHs X and a C_0–semigroup of continuous linear operators $(T(t))_{t\geq 0}$ acting in X.

In Section 3 of this paper results concerning mean ergodicity of certain classes of C_0–semigroups of operators $(T(t))_{t\geq 0}$ are established. We present sufficient conditions for various classes of C_0–semigroups to be mean ergodic (see Proposition 3, Corollary 2 and Theorem 4). In Theorem 5 we characterize the reflexivity of a complete, barrelled lcHs X with a Schauder basis via the condition that *every* equicontinuous C_0–semigroup of operators in X should be mean ergodic.

Section 4 of this paper concerns C_0–semigroups $(T(t))_{t\geq 0}$ in Grothendieck lcHs' with the Dunford–Pettis property (briefly, GDP) and their mean ergodicity. Certain classes of such C_0–semigroups are automatically uniformly continuous (see Theorem 7). Criteria are given which ensure that the dual semigroup $(T(s)')_{s\geq 0}$ is strongly continuous in X'_β (the strong dual of X). Mean ergodicity of $(T(t))_{t\geq 0}$ in Grothendieck spaces is equivalent to the subspace span$\{x' - T(s)'x' : x' \in X', s \geq 0\}$ of X' having the same closure for the weak–star topology as in X'_β; see Theorem 9. Consequently, if $(T(t))_{t\geq 0}$ is a mean ergodic, strongly continuous C_0–semigroup in a Grothendieck space and $\lim_{t\to\infty} T(t)/t = 0$ (for the topology of uniform convergence on bounded sets of X), then also $(T(s)')_{s\geq 0}$ is mean ergodic in X'_β.

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