Fréchet spaces with no infinite–dimensional Banach quotients

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It is well known that every quotient of a reflexive Banach space is also reflexive. Grothendieck [3] discovered that there are Köthe echelon spaces of order one which are Montel, hence reflexive, but have a quotient isomorphic to ℓ_1. Following Grothendieck [3, Définition 2, p.99], a lcHs X is called *totally reflexive* if every quotient of X is reflexive. Answering a question of Grothendieck [3, Probl. 9], Valdivia proved in [5, Theorem 3] that a Fréchet space X is totally reflexive if and only if it is the reduced projective limit of a sequence of reflexive Banach spaces. The aim of this paper is to give examples of non-totally reflexive, reflexive Fréchet spaces with no infinite–dimensional Banach quotients in Theorem 0.4 and non-Schwartz, Montel Fréchet spaces with no infinite–dimensional Banach quotients in Theorem 0.3. More concrete examples are given in Example 0.5. These examples answer in the negative the natural question whether every Fréchet space with a non reflexive quotient has a non reflexive Banach quotient. The question treated here is related to recent work of Bonet and Wright [2] on factorization of weakly compact operators between Banach spaces and Fréchet or (LB)–spaces. The construction uses a method to exhibit Fréchet spaces due to Albanese and Moscatelli [1]. However, the main step is accomplished in Theorem 0.1 that shows the canonical inclusion between James spaces $J_p \subset J_q, 1 < p < q < \infty$, is strictly cosingular. This results requires a careful analysis of the block basic sequences of the canonical basis of the dual J'_p of the James space J_p, and permits us to show in Theorem 0.2 that the Fréchet space $J_{p^+} = \cap_{q>p} J_q$ and the (LB)-space $J_{p^-} = \cup_{1<q<p} J_q$ have no infinite–dimensional Banach quotients. Plichko and Maslyuchenko had proved in [4] that they have no infinite–dimensional Banach subspaces. Some positive results are also included.

Theorem 0.1 *Let $1 < p < q < \infty$ and $\iota_p^q \colon J_p \hookrightarrow J_q$ denote the canonical inclusion map. Then ι_p^q and its dual map $(\iota_p^q)'$ are both strictly singular and strictly cosingular.*

Theorem 0.2 *Let $1 \leq p < \infty$. Then J_{p^+} and J_{p^-} ($p > 1$) are non-reflexive locally convex spaces with no infinite–dimensional Banach quotients.*

Theorem 0.3 *Let $1 \leq p < \infty$. If L and E are Schwartz Fréchet spaces, then the space $M(E, J_{p^+}; L)$ is a non–Schwartz, non–totally reflexive, Montel Fréchet space with no infinite–dimensional Banach quotient.*

Theorem 0.4 *Let $1 \leq p < \infty$. If L is a Schwartz Fréchet space, E is a totally reflexive Fréchet space such that the maps j_{n+1}^n and u_n are strictly cosingular, then the space $M(E, J_{p^+}; L)$ is a non–totally reflexive, reflexive Fréchet space with no infinite–dimensional Banach quotients.*

Example 0.5 (1) Let E be the nuclear Fréchet space $s = \cap_{n=1}^{\infty} \ell_1(j^n)$ of all rapidly decreasing sequences. Then, for every $1 \leq p < \infty$, the space $M(s, J_{p^+}; s)$ is a non–Schwartz, non–totally reflexive, Montel Fréchet space with no finite–dimensional Banach quotients.

(2) Let E be the totally reflexive Fréchet space $\ell_{p^+} = \cap_{q>p} \ell_q$ for some $p > 1$. Then, for every for $q \in]p, \infty)$, the space $M(\ell_{p^+}, J_{q^+}; s)$ is a non–totally reflexive, reflexive Fréchet space with no finite–dimensional Banach quotients.

REFERENCES

1. A.A. Albanese, V.B. Moscatelli, *A Method of construction of Fréchet spaces.* In: *"Functional Analysis–Selected Topics"*, P.K. Jain (Ed), Narosa Publishing House, New Delhi, 1998, pp. 1–8.
2. J. Bonet, J.D.M. Wright, *Factorization of weakly compact operators between Banach spaces and Fréchet or (LB)–spaces.* Math. Vesnik (to appear).
3. A. Grothendieck, *Sur les espaces (F) et (DF).* Summa Brasil Math. **3** (1954), 57–123.
4. A.N. Plichko, V.K. Maslyuchenko, *Quasireflexive locally convex spaces without Banach subspaces.* J. Soviet Math. **48** (1990), 307–312.
5. M. Valdivia, *A characterization of totally reflexive Fréchet spaces.* Math. Z. **200** (1989), 327–346.