# FREE GRADIENT DISCONTINUITY AND IMAGE INPAINTING

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In image restoration the term inpainting denotes the process of filling the missing information in subdomains where a given image is damaged: these domains may correspond to scratches in a camera picture, occlusion by objects, blotches in an old movie film or aging of canvas and colors in a painting ([ 3], [ 4], [ 15]).
Minimization of Blake & Zisserman functional is a variational approach to segmentation and denoising in image analysis which deals with free discontinuity, free gradient discontinuity and second derivatives: this second order functional was introduced to overcome the over-segmentation of steep gradients (ramp effect) and other drawbacks which occur in lower order models as in case of Mumford & Shah functional (see Ref. [ 18]). We refer to [ 1], [ 15], [ 9], for motivation and analysis of variational approach to image segmentation and digital image processing.
In this paper we face the inpainting problem for a monochromatic image with a variational approach: solving a Dirichlet type problem for the main part of Blake & Zisserman functional. A similar problem was studied in [ 11] with the aim of finding a segmentation of a given noisy image.
Mumford & Shah model has been adapted by several authors to the inpainting problem, but some inconvenient has been detected in this approach (see Ref. [ 15]). In the Mumford & Shah model (see Ref. [ 17], [ 18]), the preferable edge curves are those which have the shortest length, therefore it favours straight edges and it produces the emerging of artificial corners. In the Blake & Zisserman model, the presence of second derivatives smooths such corners.
About minimization of the Blake & Zisserman functional under Neumann boundary condition we refer to [ 8]. For a description of the rich list of differential, integral and geometric extremality conditions we refer to [ 9]. The results of the paper [ 11] were deeply exploited in [ 10] and [ 12] to study fine properties of local minimizers of Blake & Zisserman functional under Neuman boundary condition, particularly about their singular set related to optimal segmentation; in the present paper they are applied to the derivation and study of a variational algorithm for image inpainting.

In this paper we propose two different second order functionals $E^\delta$ and $F^\delta$ to deal with image inpainting. The two functionals respectively focus on the cases of complete or partial loss of information in a small subregion.
First we focus on the functional $E$, which is defined as follows:

$$E(K_0, K_1, v) = \int_{\Omega\setminus(K_0\cup K_1)} \left|D^2 v\right|^2 \, d\mathbf{x} + \alpha\mathcal{H}^1\left(K_0\cap\overline{\Omega}\right) + \beta\mathcal{H}^1\left((K_1\setminus K_0)\cap\overline{\Omega}\right) . \tag{1}$$

To face the inpainting problem we look for minimizers of $E^\delta = E(K_0, K_1, v) + \delta\int_\Omega |v|^2\, d\mathbf{x}$, with $\alpha, \beta, \delta > 0$, among admissible triplets $(K_0, K_1, v)$, say triplets fulfilling

$$\begin{cases} K_0\,,\, K_1 \text{ Borel subsets of } \mathbb{R}^2, \qquad K_0\cup K_1 \text{ closed,} \\ v\in C^2\left(\widetilde{\Omega}\setminus(K_0\cup K_1)\right),\ v \text{ approximately continuous in } \widetilde{\Omega}\setminus K_0, \\ v = w \text{ a.e. in } \widetilde{\Omega}\setminus\overline{\Omega}\,, \end{cases} \tag{2}$$

where $\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2$ are open sets, $\Omega$ with piecewise $C^2$ boundary and $w$ is a given function in $\widetilde{\Omega}\setminus\overline{\Omega}$.
The raw image under processing is damaged due to the presence of blotches in the set $\overline{\Omega}$: the noiseless brightness intensity $w$ of the image is known in $\widetilde{\Omega}\setminus\overline{\Omega}$ while is completely lost in the possibly disconnected set $\Omega$.
If $(K_0, K_1, u)$ is a minimizing triplet of $E^\delta$, then $u$ provides the inpainted restoration of the whole image, and $K_0\cup K_1$ can be interpreted as an optimal segmentation of the restored image: the three elements of a minimizing triplet $(K_0, K_1, u)$ play respectively the role of edges, creases and smoothly varying intensity in the region $\widetilde{\Omega}\setminus(K_0\cup K_1)$ for the segmented image.

Our result for monochromatic images is stated below in Theorem 1 in the simplified case when the image is smooth where damage does not occur.
About RGB color images, we refer to a forthcoming paper ([ 14]).

**Theorem 1.** *Let $\alpha$, $\beta$, $\delta$, $\Omega$, $\widetilde{\Omega}$ and $w$ be s.t.*

$$0 < \beta \le \alpha \le 2\beta,\ \ \delta > 0 \tag{3}$$

$$\Omega \subset\subset \widetilde{\Omega} \subset\subset \mathbb{R}^2\,, \tag{4}$$

$$\Omega \textit{ is an open set with piecewise } C^2 \textit{ boundary } \partial\Omega\,,\ \widetilde{\Omega} \textit{ is an open set,} \tag{5}$$

$$w \textit{ has a } C^2(\widetilde{\Omega}) \textit{ extension which fulfils } \ D^2 w \in L^\infty(\widetilde{\Omega})\,. \tag{6}$$

*Then there exists a triplet $(C_0, C_1, u)$ which minimizes the functional*

$$E^\delta(K_0, K_1, v) := E(K_0, K_1, v) + \delta \int_\Omega |v|^2\, d\mathbf{x} \tag{7}$$

*with finite energy, among admissible triplets $(K_0, K_1, v)$ according to (1), (2),*
*Moreover any minimizing triplet $(K_0, K_1, v)$ fulfils:*

$$K_0 \cap \overline{\Omega} \textit{ and } K_1 \cap \overline{\Omega} \textit{ are } (\mathcal{H}^1, 1) \textit{ rectifiable sets,} \tag{8}$$

$$\mathcal{H}^1(K_0 \cap \overline{\Omega}) \ = \ \mathcal{H}^1(\overline{S_v})\,, \quad \mathcal{H}^1(K_1 \cap \overline{\Omega}) \ = \ \mathcal{H}^1(\overline{S_{\nabla v}} \setminus S_v)\,, \tag{9}$$

$$\begin{cases} v \in GSBV^2(\widetilde{\Omega}),\ \textit{hence} \\ v \textit{ and } \nabla v \textit{ have well defined two-sided traces, finite } \mathcal{H}^1 \textit{ a.e. on } K_0 \cup K_1\,, \end{cases} \tag{10}$$

*where $S_v$ and $S_{\nabla v}$ respectively denote the singular sets of $v$ and $\nabla v$.*

The main result of this paper is in Theorem 2 (which is not reported here): the statement is quite technical but it is a more useful tool than Theorem 1, since it deals with discontinuity and gradient discontinuity in $\widetilde{\Omega} \setminus \Omega$ of the given raw image $w$ to be processed, together with some additional noisy information denoted by $g$ in a Borel subset $\Omega \setminus U$, where

$$U \subset\subset \Omega \subset\subset \widetilde{\Omega}. \tag{11}$$

Theorem 2 provides the existence of minimizers for the second functional proposed in this paper, which is labeled with $F^\delta$ and deals with the noisy part of the image adding a fidelity term to the functional $E^\delta$. Precisely, we set

$$F^\delta(K_0, K_1, v) = E^\delta(K_0, K_1, v) + \mu \int_{\Omega\setminus U} |v - g|^2\ d\mathbf{x} \tag{12}$$

and we look for minimizers of $F^\delta(K_0, K_1, v)$ among triplets $(K_0, K_1, v)$ verifying (2). We apply direct methods of Calculus of Variations to functional (12) by proving the partial regularity for solutions of a weak version $\mathcal{F}^\delta$ of (12).

We emphasize that if $(K_0, K_1, v)$ is a minimizing triplet of $F^\delta$ than $v$ fulfils the Euler equations

$$\Delta^2 v + \mu(v - g) = 0 \qquad \text{in } \ \Omega \setminus (\overline{U} \cup K_0 \cup K_1), \tag{13}$$

$$\Delta^2 v + \delta v = 0 \qquad \text{in } \ U \setminus (K_0 \cup K_1) \tag{14}$$

together with many kind of integral and geometric relationships as like as minimizing triplet of Blake & Zisserman functional for image segmentation (see [ 9], [ 12]).
To achieve the existence of minimizing triplets of $F^\delta$, inspired by the seminal paper of De Giorgi and Ambrosio [ 16], we introduce a relaxed functional: the weak Blake & Zisserman functional for inpainting $\mathcal{F}^\delta(v)$. The idea is to deal with a simpler object, just depending on the function $v$, and then to recover the set of jumps $K_0$ and creases $K_1 \setminus K_0$ by taking respectively the discontinuity set $\overline{S_v}$ and $\overline{S_{\nabla v}} \setminus S_v$. The functional class where we set the problem is given by second order generalized functions with special bounded variation: say $GSBV^2(\widetilde{\Omega})$. The class $GSBV^2(\widetilde{\Omega})$ is the right functional setting, more appropriate than $BH(\widetilde{\Omega})$ (bounded hessian functions whose second derivatives are Radon measure). Indeed $BH$

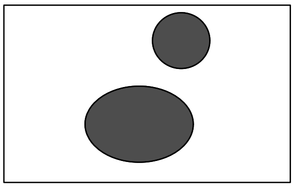

Figure 1. Theorem 1: the image domain is the rectangle $\widetilde{\Omega}$, the blotches $\Omega \subset\subset \widetilde{\Omega}$ with complete loss of information are the black region $\overline{\Omega}$.

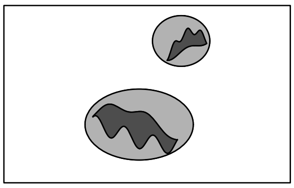

Figure 2. Theorem 2: the image domain is the rectangle $\widetilde{\Omega}$, the blotches $\Omega \subset\subset \widetilde{\Omega}$ with some loss of information, complete loss of information in the black region $U$, the partially damaged image is given in the gray region $\Omega \setminus U$.

functions in two variables are continuous with integrable gradient; nevertheless $BH$ contains too much irregular functions: admissible functions may have gradient with nontrivial Cantor part.
In this framework compactness and lower semicontinuity Theorems 8 and 10 of [ 7] give the existence of minimizers for the relaxed functional $\mathcal{F}^{\delta}(v)$. The results of Theorems 1 and 2 are achieved by showing partial regularity of the obtained weak solution with penalized Dirichlet datum. The novelty here consists in the regularization at the boundary for a free gradient discontinuity problem with Dirichlet datum (in the set $\partial\Omega$) or transmission condition (in the set $\partial U$).

A numerical scheme, based on the theory of $\Gamma$-convergence, as in [ 2] and [ 5], the convergence analysis and its implementation are contained in a forthcoming paper [ 13].

We conclude by showing some pictures obtained in numerical experiments which exploit the variational approximation of the functional (12): Figures 3 and 4 where the inpainting algorithm removes masks or overlapping text.

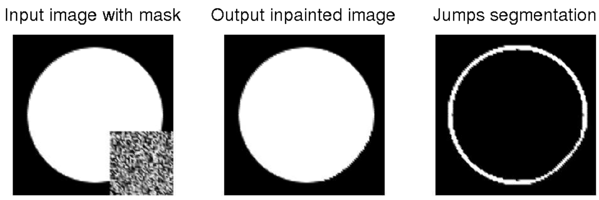


Figure 3. Inpainting of a circle without introducing artificial corners.


Figure 4. Text removal.

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